Computing Abelian Covers and Abelian Runs

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Abstract. Two strings $u$ and $v$ are said to be Abelian equivalent if $u$ is a permutation of the characters of $v$. We introduce two new regularities on strings w.r.t. Abelian equivalence, called Abelian covers and Abelian runs, which are generalizations of covers and runs of strings, respectively. We show how to determine in $O(n)$ time whether or not a given string $w$ of length $n$ has an Abelian cover. Also, we show how to compute an $O(n^2)$-size representation of (possibly exponentially many) Abelian covers of $w$ in $O(n^2)$ time. Moreover, we present how to compute all Abelian runs in $w$ in $O(n^2)$ time, and state that the maximum number of all Abelian runs in a string of length $n$ is $\Omega(n^2)$.

Keywords: Abelian equivalence on strings, Parikh vectors, Abelian repetitions, covers of strings, string algorithms

1 Introduction

The study of Abelian equivalence of strings dates back to at least the early 60’s, as seen in the paper by Erdős [6]. Two strings $u, v$ are said to be Abelian equivalent if $u$ is a permutation of the characters appearing in $v$. For instance, strings $aabba$ and $baaba$ are Abelian equivalent. Abelian equivalence of strings has attracted much attention and has been studied extensively in several contexts.

A variant of the pattern matching problem called the jumbled pattern matching problem is to determine whether there is a substring of an input text string $w$ that is Abelian equivalent to a given pattern string $p$. There is a folklore algorithm to solve this problem in $O(n + m + \sigma)$ time using $O(\sigma)$ space, where $n$ is the length of $w$, $m$ is the length of $p$, and $\sigma$ is the alphabet size. Assuming $m \leq n$ and all characters appear in $w$, the algorithm runs in $O(n)$ time and $O(\sigma)$ space. The indexed version of the jumbled pattern matching problem is more challenging, where the task is to preprocess an input text string $w$ so that, given a query pattern string $p$, we can quickly determine whether or not there is a substring of $w$ that is Abelian equivalent to $p$. For binary strings, there exists a data structure which occupies $O(n)$ space and answers the above query in $O(1)$ time. Burcsi et al. [2] and Moosa and Rahman [16] independently developed an $O(n^2/\log n)$-time algorithm to construct this data structure, and later Gagie et al. [9] showed an improved $O(n^2/\log^2 n)$-time algorithm. Very recently, Hermelin et al. [10] proposed an $n^2/2^{O(\log n/\log \log n)}\cdot n^3$-time solution to the problem for binary strings. For any constant-size alphabets, Kociumaka et al. [12] showed an algorithm that requires $O(n^2 \log^2 \log n/\log n)$ preprocessing time and $O((\log n/\log \log n)^{2\sigma-1})$ query time. Amir et al. [1] showed lower bounds on the indexing version of the jumbled pattern matching problem under a 3SUM-hardness assumption.

Abelian periodicity of strings has also been extensively studied in string algorithmics. A string $w$ is said to have a full Abelian period if $w$ is a concatenation $w_1 \cdots w_k$ of $k$ Abelian equivalent strings $w_1, \ldots, w_k$ with $k \geq 2$, and the length of $w_1$ is called a full Abelian period of $w$. A string $w$ is said to have an Abelian period if $w = yz$. 
where \( y \) has some full Abelian period \( d \), and \( z \) is a non-empty string shorter than \( d \) such that the number of each character \( a \) contained in \( z \) is no more than that contained in the prefix \( y[1..d] \) of length \( d \) of \( y \). A string \( w \) is said to have a weak Abelian period \( d \) if \( w = xy \), where \( y \) has some Abelian period \( d \), and \( x \) is a non-empty string shorter than \( d \) such that the number of each character \( a \) contained in \( x \) is no more than that contained in the prefix \( y[1..d] \) of length \( d \) of \( y \). Fici et al. \[7\] proposed an \( O(n \log \log n) \)-time algorithm to compute all full Abelian periods and \( O(n^2) \)-time algorithm to compute all Abelian periods for a given string of length \( n \). Recently, Kociumaka et al. \[8\] showed an optimal \( \Theta(n) \)-time algorithm to compute all full Abelian periods, and an improved \( O(n(\log \log n + \log \sigma)) \)-time algorithm to compute all Abelian periods, where \( \sigma \) is the alphabet size. Fici et al. \[8\] presented an \( O(n^2\sigma) \)-time algorithm to compute all weak Abelian periods, and later Crochemore et al. \[8\] gave an improved \( O(n^2) \)-time solution to the problem.

In the field of word combinatorics, Erdős \[6\] posed a question whether there exists an infinitely long string which contains no Abelian squares. A substring \( s \) of a string \( w \) is called an Abelian cover if \( s = s_1 s_2 \) such that \( s_1 \) and \( s_2 \) are Abelian equivalent. Entringer et al. \[5\] proved that any infinite word over a binary alphabet contains arbitrary long Abelian squares. On the other hand, Pleasants \[18\] showed a construction of an infinitely long string which contains no Abelian squares over an alphabet of size 5, and later, Keränen \[11\] showed a construction over an alphabet of size 4. An interesting question in the field of string algorithmics is how efficiently we can compute the Abelian repetitions that occur in a given string of finite length \( n \). Cummings and Smyth \[3\] presented an algorithm to compute all Abelian squares in \( O(n^2) \) time. They also showed that there exist \( \Omega(n^2) \) Abelian squares in a string of length \( n \). Crochemore et al. \[8\] showed another \( O(n^2) \)-time algorithm to compute all squares in a string of length \( n \).

In this paper, we introduce two new regularities on strings w.r.t. Abelian equivalence, called Abelian covers and Abelian runs, which are generalizations of covers \[15\] and runs \[14\] of strings, respectively, and we propose non-trivial algorithms to compute these new string regularities. A set \( C \) of intervals is called an Abelian cover of a string \( w \) if the substrings corresponding to the intervals in \( C \) are all Abelian equivalent, and every position in \( w \) is contained in at least one interval in \( C \). We show that, given a string \( w \) of length \( n \), we can determine whether or not \( w \) has an Abelian cover in optimal \( O(n) \) time. Also, we present an \( O(n^2) \)-time algorithm to compute an \( O(n^2) \)-size representation of all (possibly exponentially many) Abelian covers of \( w \). A substring \( s \) of \( w \) is said to be an Abelian run of \( w \) if \( s \) is a maximal substring which has a weak Abelian period. As a direct consequence from the result by Cummings and Smyth \[3\], it is shown that the maximum number of all Abelian runs in a string of length \( n \) is \( \Omega(n^2) \). Then, we propose an \( O(n^2) \)-time algorithm to compute all Abelian runs in a given string of length \( n \).

2 Preliminaries

Let \( \Sigma = \{c_1, \ldots, c_\sigma\} \) be an ordered alphabet. We assume that for each \( c_i \in \Sigma \), its rank \( i \) in \( \Sigma \) is already known and can be computed in constant time. An element of \( \Sigma^* \) is called a string. The length of a string \( w \) is denoted by \(|w|\). The empty string \( \varepsilon \) is the string of length 0, namely, \(|\varepsilon| = 0\). For a string \( w = xyz \), strings \( x \), \( y \), and \( z \) are called a prefix, substring, and suffix of \( w \), respectively. The \( i \)-th character of a string \( w \) of length \( n \) is denoted by \( w[i] \) for \( 1 \leq i \leq n \). For \( 1 \leq i \leq j \leq n \), let \( w[i..j] = w[i] \cdots w[j] \),
i.e., $w[i..j]$ is the substring of $w$ starting at position $i$ and ending at position $j$ in $w$. For convenience, let $w[i..j] = \varepsilon$ if $j < i$. For any $0 \leq i \leq n$, strings $w[1..i]$ and $w[i..n]$ are called prefixes and suffixes of $w$, respectively.

For any string $w$ of length $n \geq 2$, a set $I = \{[b_1, e_1], \ldots, [b_I, e_I]\}$ of intervals is called a cover of $w$ if $\bigcup_{1 \leq k \leq I} [b_k, e_k] = \{1, n\}$ and $[b_k, e_k] \neq \{1, n\}$ for every $1 \leq k \leq |I|$. Whenever we write $C = \{[b_1, e_1], \ldots, [b_C, e_C]\}$ for a cover $C$ of a string $w$, then we assume that $b_j < b_{j+1}$ for all $1 \leq k < |C|$. Two strings $v, w \in \Sigma^*$ are said to be Abelian equivalent if $v$ is a permutation of the characters in $w$. A Parikh vector $[\sigma]$ of a string $w \in \Sigma^*$, denoted $P_w$, is an array of length $\sigma$ such that for any $1 \leq i \leq \sigma$, $P_w[i]$ stores the number of occurrences of character $c_i$ in $w$. Let $\leq$ be a partial order of Parikh vectors $P_v$ and $P_w$ for any strings $v, w \in \Sigma^*$ such that

\[
P_v = P_w \text{ if } P_v[i] = P_w[i] \text{ for all } 1 \leq i \leq \sigma, \quad \text{and} \quad P_v < P_w \text{ if } P_v \neq P_w \text{ and } P_v[i] \leq P_w[i] \text{ for all } 1 \leq i \leq \sigma.
\]

For instance, for strings $v = \text{aababc}$ and $w = \text{baba}$ over an ordered alphabet $\Sigma = \{a, b, c\}$, $P_v = [3, 2, 1]$ and $P_w = [2, 2, 0]$, and therefore $P_w < P_v$. Clearly, strings $v, w$ are Abelian equivalent if $P_v = P_w$. For any two strings $v, w \in \Sigma^*$, let $P_v \oplus P_w = P_{vw}$, namely, $(P_v \oplus P_w)[i] = P_v[i] + P_w[i]$ for each $1 \leq i \leq \sigma$.

A cover $C = \{[b_1, e_1], \ldots, [b_C, e_C]\}$ of a string $w$ is called an Abelian cover of $w$ if $P_{w[b_1..e_1]} = P_{w[b_j..e_j]}$ for all $1 < j \leq |C|$. Clearly $e_j - b_j = e_1 - b_1$ holds for all $1 < j \leq |C|$. The length and size of an Abelian cover $C = \{[b_1, e_1], \ldots, [b_C, e_C]\}$ of a string are $e_1 - b_1 + 1$ and $|C|$, respectively. See Figure 1 for examples of Abelian covers of a string.

A non-empty substring $u$ of a string $w$ is called an Abelian repetition with period $d$ if $|u|$ is a multiple of an integer $d$ ($1 \leq d \leq \frac{|u|}{2}$) and $P_u[(k-1)d+1..kd] = P_u[kd+1..(k+1)d]$ for all $1 \leq k < \frac{|u|}{d}$. If $d = \frac{|w|}{2}$, then $u$ is called an Abelian square of $w$. A substring $w[i..j]$ of a string $w$ is called a maximal Abelian repetition of $w$ if $w[i..j]$ is a non-extensible Abelian repetition with period $d$ in $w$, namely, if $w[i..j]$ is an Abelian repetition satisfying (1) $P_{w[i..i-d..i-1]} \neq P_{w[i..i-d+1..i]}$ or $i - d < 0$ and (2) $P_{w[j-d+1..j]} \neq P_{w[j+1..j+d]}$ or $j + d > n$. A substring $w[i..j] + h'$ of a string $w$ is called an Abelian run.
Abelian cover of a given string \( w \).

In this subsection, we consider Problem 1 of determining whether there exists an Abelian cover of size 2 of a given string \( w \) of length \( n \). Note that there exists an infinite sequence of strings over a binary alphabet which have no Abelian covers (e.g., \( a^{n-1}b \) has no Abelian covers), and therefore Problem 1 is of interest. The following lemma is a key to our solution to the problem.

**Lemma 4.** If there exists an Abelian cover \( C = \{ [b_1, e_1], \ldots, [b_{|C|}, e_{|C|}] \} \) of arbitrary size for a string \( w \), then there exists an Abelian cover \( C' \) of size exactly 2 for \( w \).

**Proof.** It is clear when \( |C| = 2 \). Consider the case where \( |C| \geq 3 \). Since \( C \) is an Abelian cover of \( w \), \( P_{w|b_1, e_1|} = P_{w|b_{|C|}, e_{|C|}|} \). This implies that \( P_{w|b_1, e_1|+1, b_{|C|}-1} + P_{w|b_{|C|}, e_{|C|}|} \). Therefore, \( C' = \{ [b_1, b_{|C|} - 1], [e_1 + 1, |w|] \} = \{ [1, b_{|C|} - 1], [e_1 + 1, |w|] \} \) is an Abelian cover of size 2 of \( w \). (See also Figure 3.)

Using the above lemma, we obtain the following.

**Theorem 5.** Given a string \( w \) of length \( n \), we can determine whether \( w \) has an Abelian cover or not in \( O(n) \) time with \( O(\sigma) \) working space.

**Figure 2.** String `aabbacbaabbb` over a ternary alphabet \( \Sigma = \{a, b, c\} \) has Abelian runs 
\( (0, 1, 2, 1, 0), (0, 3, 4, 1, 0), (0, 7, 8, 1, 0), (0, 9, 10, 1, 0), (0, 11, 14, 1, 0) \) of period 1, Abelian runs 
\((1, 8, 11, 2, 1), (1, 2, 5, 2, 0) \) of period 2, and an Abelian run \((0, 7, 12, 3, 2) \) of period 3.

of \( w \) if \( w[i..j] \) is a maximal Abelian repetition with period \( d \) of \( w \) and \( h, h' \geq 0 \) are the largest integers satisfying \( P_{w|i−h..i−1} \preceq P_{w|i..i+d} \) and \( P_{w|i+1..j+h'} \preceq P_{w|j−d..j} \), respectively. Each Abelian run \( w[i−h..j+h'] \) of period \( d \) in \( w \) is represented by a 5-tuple \( (h, i, j, d, h') \), where \( h \) and \( h' \) are called the left hand and the right hand of the Abelian run, respectively. See Figure 2 for examples of Abelian runs in a string.

In this paper, we consider the following problems.

**Problem 1 (Abelian cover existence).** Given a string \( w \), determine whether or not \( w \) has an Abelian cover.

**Problem 2 (All Abelian covers).** Given a string \( w \), compute all Abelian covers of \( w \).

**Problem 3 (All Abelian runs).** Given a string \( w \), compute all Abelian runs in \( w \).

### 3 Algorithms

In this section, we present our algorithms to solve the problems stated in the previous section. For simplicity, we assume that all characters in \( \Sigma \) appear in a given string \( w \) of length \( n \), which implies \( \sigma \leq n \).

#### 3.1 Abelian cover existence

In this subsection, we consider Problem 1 of determining whether there exists an Abelian cover of a given string \( w \) of length \( n \). Note that there exists an infinite sequence of strings over a binary alphabet which have no Abelian covers (e.g., \( a^{n−1}b \) has no Abelian covers), and therefore Problem 1 is of interest. The following lemma is a key to our solution to the problem.

**Lemma 4.** If there exists an Abelian cover \( C = \{ [b_1, e_1], \ldots, [b_{|C|}, e_{|C|}] \} \) of arbitrary size for a string \( w \), then there exists an Abelian cover \( C' \) of size exactly 2 for \( w \).

**Proof.** It is clear when \( |C| = 2 \). Consider the case where \( |C| \geq 3 \). Since \( C \) is an Abelian cover of \( w \), \( P_{w|b_1, e_1|} = P_{w|b_{|C|}, e_{|C|}|} \). This implies that \( P_{w|b_1, e_1|+1, b_{|C|}-1} + P_{w|b_{|C|}, e_{|C|}|} \). Therefore, \( C' = \{ [b_1, b_{|C|} - 1], [e_1 + 1, |w|] \} = \{ [1, b_{|C|} - 1], [e_1 + 1, |w|] \} \) is an Abelian cover of size 2 of \( w \). (See also Figure 3.)

Using the above lemma, we obtain the following.

**Theorem 5.** Given a string \( w \) of length \( n \), we can determine whether \( w \) has an Abelian cover or not in \( O(n) \) time with \( O(\sigma) \) working space.
find such a prefix and a suffix, for each \( \ell \) to find a prefix and a suffix of the same length given string \( w \).

**Proof.** By Lemma 4 of deciding whether there exists an Abelian cover of a given string \( w \) reduces to finding an Abelian cover of size 2 of \( w \). Therefore, it suffices to find a prefix and a suffix of the same length \( \ell \) such that \( P_{w[1..\ell]} = P_{w[n-\ell+1..n]} \). To find such a prefix and a suffix, for each \( 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \) in increasing order, we maintain an invariant \( d_j \) which represents the number of entries of \( P_{w[1..\ell]} \) and \( P_{w[n-\ell+1..n]} \) whose values differ, i.e.,

\[
d_j = \{ k \mid P_{w[1..\ell]}[k] \neq P_{w[n-\ell+1..n]}[k], 1 \leq k \leq \sigma \}.
\]

Clearly \( w \) has an Abelian cover of size 2 iff \( d_j = 0 \) for some \( j \). For any \( 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \), let \( w[j] = c_s \) and \( w[n-j+1] = c_t \). We can update \( P_{w[1..j-1]} \) (resp. \( P_{w[n-j..n]} \)) to \( P_{w[1..j]} \) (resp. \( P_{w[n-j+1..n]} \)) in \( O(1) \) time, increasing the value stored in the \( s \)th entry (resp. the \( t \)th entry) by 1. Also, \( d_j \) can be computed in \( O(1) \) time from \( d_{j-1}, P_{w[1..j-1][s]}, P_{w[1..j][s]}, P_{w[n-j..n][t]}, \) and \( P_{w[n-j+1..n][t]} \). Hence, the algorithm runs in a total of \( O(n) \) time. The extra working space of the algorithm is \( O(\sigma) \), due to the two Parikh vectors we maintain.

The following corollary is immediate from Lemma 4 and Theorem 5.

**Corollary 6.** We can compute the longest Abelian cover of a given string of length \( n \) in \( O(n) \) time with \( O(\sigma) \) working space, if it exists.

### 3.2 All Abelian covers

In this subsection, we consider Problem 2 of computing all Abelian covers of a given string \( w \) of length \( n \). Note that the number of all Abelian covers of a string can be exponentially large w.r.t. \( n \). For instance, string \( a^n \) has \( \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{n-1} 2^{n-k-1} \) Abelian covers of length at least \( \left\lceil \frac{n}{2} \right\rceil \). This is because, for any \( k \geq \left\lceil \frac{n}{2} \right\rceil \), the union of \( \{[1,k], [n-k+1,n]\} \) and any subset of \( \{[2,k+1],[3,k+2],\ldots,[n-k,n-1]\} \) is an Abelian cover of length \( k \) for \( a^n \). Therefore, we consider to compute a “compact” representation of all Abelian covers of a given string.

**Theorem 7.** Given a string \( w \) of length \( n \), we can compute an \( O(n^2) \)-size representation of all Abelian covers of \( w \) in \( O(n^2) \) time and \( O(n) \) working space. Given a set \( I \) of \( s \) intervals sorted by the beginning positions of the intervals, the representation allows us to check if \( I \) is an Abelian cover of \( w \) in \( O(s) \) time.

**Proof.** For each \( 1 \leq \ell \leq n-1 \), we compute a subset \( S_\ell \) of positions in \( w \) such that \( S_\ell = \{ i \mid P_{w[i,i+\ell-1]} = P_{w[1..\ell]}, 1 \leq i \leq n-\ell+1 \} \). Then, there exists an Abelian cover
of length $\ell$ for $w$ if the distance between any two adjacent positions in $S_\ell$ is at most $\ell$. If $S_\ell$ satisfies the above condition, then we represent $S_\ell$ as a bit vector $B_\ell$ of length $n$ such that $B_\ell[i] = 1$ if $i \in S_\ell$, and $B_\ell[i] = 0$ otherwise. If $S_\ell$ does not satisfy the above condition, then we discard it. Now, given a set $I$ of $s$ intervals sorted by the beginning positions of the intervals, we first check if $I$ is a cover of $w$ and if each interval is of equal length $\ell$ in a total of $O(s)$ time. If $I$ satisfies both conditions, then we can check if $I$ is a subset of $S_\ell$ in $O(s)$ time, using the bit vector $B_\ell$. Using a similar method to Theorem 5, for each $1 \leq \ell \leq n - 1$, $S_\ell$ and its corresponding bit vector $B_\ell$ can be computed in $O(n)$ time. Hence, the overall time complexity of the algorithm is $O(n^2)$. The working space (excluding the output) is $O(n)$, since $|S_\ell| = O(n)$ for any $\ell$ and $\sigma = O(n)$. \qed

Given a set $I$ of $s$ intervals, a naive algorithm to check whether $I$ is an Abelian cover of length $\ell$ requires $O(sl)$ time. Therefore, the solution of Theorem 7 with $O(s)$ query time is more efficient than the naive method.

### 3.3 All Abelian runs

In this subsection, we consider Problem 3 of computing all Abelian runs in a given string $w$ of length $n$. We follow and extend the results by Cummings and Smyth [4] on the maximum number of all maximal Abelian repetitions in a string, and an algorithm to compute them. We firstly consider a lower bound on the maximum number of Abelian runs in a string.

**Lemma 8 ([4]).** String $(aababbaba)^n$ of length $8n$ has $\Theta(n^2)$ maximal Abelian repetitions (in fact maximal Abelian squares).

Since the number of Abelian runs in a string is equal to that of maximal Abelian repetitions in that string, the following theorem is immediate:

**Theorem 9.** The maximum number of Abelian runs in a string $w$ of length $n$ is $\Omega(n^2)$.

Next, we show how to compute all Abelian runs in a given string.

**Theorem 10.** Given a string $w$ of length $n$, we can compute all Abelian runs in $w$ in $O(n^2)$ time and space.

**Proof.** We firstly compute all Abelian squares in $w$ using the algorithm proposed by Cummings and Smyth [4]. For each $1 \leq i \leq n$, we compute a set $L_i$ of integers such that

$$L_i = \{ j \mid P_{w[i-j..i]} = P_{w[i+1..i+j+1]}, 0 \leq j \leq \min\{i, n - i\} \}.$$  

Note that substring $w[i - \ell..i + \ell + 1]$ is an Abelian square centered at position $i$ iff $\ell \in L_i$. After computing all $L_i$’s, we store them in a two dimensional array $L$ of size $[\frac{n}{2}] \times n - 1$ such that $L[i, i] = 1$ if $\ell \in L_i$ and $L[i, i] = 0$ otherwise. All entries of $L$ are initialized unmarked. Then, for each $1 \leq \ell \leq n - 1$, all maximal Abelian repetitions of period $\ell$ can be computed by in $O(n)$ time, as follows. We scan the $\ell$th row of $L$ from left to right for increasing $i = 1, \ldots, n - 1$, and if we encounter an unmarked entry $(\ell, i)$ such that $L[\ell, i] = 1$, then we compute the largest non-negative integer $k$ such that $L[\ell, i + p\ell + 1] = 1$ for all $1 \leq p \leq k$ in $O(k)$ time, by skipping every $\ell - 1$ entries in between. This gives us a maximal Abelian repetitions with period $\ell$.
starting at position \( i - \ell + 1 \) and ending at position \( i + (k + 1)\ell \). After computing the largest integer \( k \), we mark the entries \( L[\ell, i + p\ell + 1] \) for all \(-1 \leq p \leq k\) in \( O(k) \) time. Since each unmarked entry of the \( \ell \)th row is marked at most once and is accessed by a constant number of times, and since the above procedure starts only from unmarked entries, it takes a total of \( O(n) \) time for each \( \ell \). Therefore, this takes a total of \( O(n^2) \) time for all \( 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \). (See also Figure 4 for a concrete example of the two dimensional array \( L \) and how to compute all maximal Abelian repetitions from \( L \).

What remains is how to compute the left and right hands of each maximal Abelian runs. If we compute the left and right hands naively for all the maximal Abelian repetitions, then it takes a total of \( O(n^3) \) time due to Theorem 9. To compute the left and right hands in a total of \( O(n^2) \) time, we use the following property on Abelian repetitions: For each \( 1 \leq i \leq n \), let \( P_i \) be the set of positive integers such that for each \( \ell \in P_i \) there exists a maximal Abelian repetition whose period is \( \ell \) and beginning position is \( i - \ell + 1 \). For any \( 1 \leq j \leq |P_i| \), let \( \ell_j \) denote the \( j \)th smallest element of \( P_i \). We process \( \ell_j \) in increasing order of \( j = 1, \ldots, |P_i| \). Let \( h_j \) denote the left hand of the Abelian run that is computed from the maximal Abelian repetition whose period is \( \ell_j \) and beginning position is \( i - \ell_j + 1 \). For any \( 1 \leq j < |P_i| \), assume that we have computed the length of the left hand \( h_{j-1} \) of the maximal Abelian repetition beginning at position \( i - \ell_{j-1} + 1 \). We are now computing the left hand \( h_j \) of the next Abelian run. There are two cases to consider:

1. If \( j = 1 \) or \( \ell_{j-1} + h_{j-1} \leq \ell_j \), then we compute the left hand \( h_j \) of the maximal Abelian repetition beginning at position \( i - \ell_j + 1 \), by comparing the Parikh vector \( P_{w[i-\ell_j-k..i-\ell_j]} \) for increasing \( k \) from 0 up to \( h_j + 1 \), with the Parikh vector \( P_{w[i-\ell_j+1..i]} \). This takes \( O(h_j) \) time. (See also Figure 5).

2. If \( \ell_{j-1} + h_{j-1} > \ell_j \), then

\[
P_{w[i-\ell_{j-1}-h_{j-1}+1..i-\ell_j]} \preceq P_{w[i-\ell_{j-1}-h_{j-1}+1..i-\ell_j-1]} \preceq P_{w[i-\ell_{j-1}+1..i]} \preceq P_{w[i-\ell_j+1..i]}.
\]

This implies that \( h_j \geq \ell_{j-1} + h_{j-1} - \ell_j \). We can compute \( P_{w[i-\ell_{j-1}-h_{j-1}+1..i-\ell_j]} \) from \( P_{w[i-\ell_{j-1}-h_{j-1}+1..i-\ell_j-1]} \) in \( O(\ell_j - \ell_{j-1}) \) time. Then, we compute the left hand \( h_j \).
Let \( J^1_i \) and \( J^2_i \) be the disjoint subsets of \([1, |P|]\) such that \( j \in J^1_i \) if \( \ell_j \in P \) corresponds to Case 1 and \( j \in J^2_i \) if \( \ell_j \in P \) corresponds to Case 2. Then, the summations \( \sum_{j \in J^1_i}(h_j), \sum_{j \in J^2_i}(\ell_j - \ell_j - 1), \) and \( \sum_{j \in J^2_i}(h_j + \ell_j - \ell_j - 1 - h_j - 1) \) corresponding to the time costs for Cases 1 and 2 are all bounded by \( O(n) \). Therefore, it takes a total of \( O(n) \) time to compute the left hands of all Abelian runs that correspond to \( P \), and the right hands can be computed similarly. Hence, it takes a total of \( O(n^2) \) time to compute all Abelian runs in \( w \). The working space of the algorithm is dominated by the two dimensional array \( L \), which takes \( O(n^2) \) space.

\[ \square \]

### 4 Conclusions and future work

Abelian regularities on strings were initiated by Erdős [6] in the early 60’s, and since then they have been extensively studied in Stringology. In this paper, we introduced new regularities on strings with respect to Abelian equivalence on strings, which we call Abelian covers and Abelian runs. Firstly, we showed an optimal \( O(n) \)-time \( O(\sigma) \)-space algorithm to determine whether or not a given string \( w \) of length \( n \) over an alphabet of size \( \sigma \) has an Abelian cover. As a consequence of this, we can compute the longest Abelian cover of \( w \) in \( O(n) \)-time. Secondly, we showed an \( O(n^2) \)-time algorithm to compute an \( O(n^2) \)-space representation of all (possibly exponentially many) Abelian covers of a string of length \( n \). Thirdly, we presented an \( O(n^2) \)-time
algorithm to compute all Abelian runs in a string of length \( n \). We also remarked that the maximum number of Abelian runs in a string of length \( n \) is \( \Omega(n^2) \).

Our future work includes the following:

- The algorithm of Theorem \( \square \) allows us to compute a shortest Abelian cover of a given string of length \( n \) in \( O(n^2) \) time. Can we compute a shortest Abelian cover in \( o(n^2) \) time?
- The algorithm of Theorem \( \blacksquare \) requires \( \Theta(n^2) \) time to compute all Abelian runs of a given string \( w \) of length \( n \). This is due to the two dimensional array \( L \) of \( \Theta(n^2) \) space. Can we compute all Abelian runs in \( w \) in optimal \( O(n + r) \) time, where \( r \) is the number of Abelian runs in \( w \)?

References