On Bijective Variants of the Burrows-Wheeler Transform

Manfred Kufleitner

Universität Stuttgart
Germany

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Burrows-Wheeler transform

- Last column of sorted list of rotations
  + index of input within this list

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- Example $w = cbccabaacb$

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Rotations

$c\ b\ c\ c\ a\ b\ a\ a\ c\ b$
$b\ c\ b\ c\ c\ a\ b\ a\ a\ c$
$c\ b\ c\ b\ c\ c\ a\ b\ a\ a$
$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$
$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
$b\ a\ a\ c\ b\ c\ b\ c\ c\ a$
$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$
$c\ a\ b\ a\ a\ c\ b\ c\ b\ c$
$c\ c\ a\ b\ a\ a\ c\ b\ c\ b$
$b\ c\ c\ a\ b\ a\ a\ c\ b\ c$

Burrows-Wheeler transform

- Last column of sorted list of rotations
  $+$ index of input within this list
- Example $w = cbccabaacb$

Rotations	Sorted list
$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$	$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$
$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$	
$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$	
$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$	
$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$	
$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$	
$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$	
$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$	
$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$	
$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$	

Burrows-Wheeler transform

- Last column of sorted list of rotations
  $+$ index of input within this list
- Example $w = cbccabaacb$

Rotations

$c\ b\ c\ c\ a\ b\ a\ a\ c\ b$
$b\ c\ b\ c\ c\ a\ b\ a\ a\ c$
$c\ b\ c\ b\ c\ c\ a\ b\ a\ a$
$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$
$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
$b\ a\ a\ c\ b\ c\ b\ c\ c\ a$
$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$
$c\ a\ b\ a\ a\ c\ b\ c\ b\ c$
$c\ c\ a\ b\ a\ a\ c\ b\ c\ b$
$b\ c\ c\ a\ b\ a\ a\ c\ b\ c$

Sorted list

$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$

Burrows-Wheeler transform

- Last column of sorted list of rotations
  - + index of input within this list
- Example $w = cbccabaacb$

Rotations

$c\ b\ c\ c\ a\ b\ a\ a\ c\ b$
$b\ c\ b\ c\ c\ a\ b\ a\ a\ c$
$c\ b\ c\ b\ c\ c\ a\ b\ a\ a$
$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$
$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
$b\ a\ a\ c\ b\ c\ b\ c\ c\ a$
$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$
$c\ a\ b\ a\ a\ c\ b\ c\ b\ c$
$c\ c\ a\ b\ a\ a\ c\ b\ c\ b$
$b\ c\ c\ a\ b\ a\ a\ c\ b\ c$

Sorted list

$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$
$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$

Burrows-Wheeler transform

- Last column of sorted list of rotations

  $+$ index of input within this list
- Example $w = cbccabaacb$

Rotations	Sorted list
c b c c a b a a c b	*a a c b c b c c a b*
b c b c c a b a a c	*a b a a c b c b c c*
c b c b c c a b a a	*a c b c b c c a b a*
a c b c b c c a b a	*b a a c b c b c c a*
a a c b c b c c a b	*b c b c c a b a a c*
b a a c b c b c c a	*b c c a b a a c b c*
a b a a c b c b c c	*c a b a a c b c b c*
c a b a a c b c b c	*c b c b c c a b a a*
c c a b a a c b c b	*c b c c a b a a c b*
b c c a b a a c b c	*c c a b a a c b c b*

Burrows-Wheeler transform

- Last column of sorted list of rotations
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Rotations	Sorted list
$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$	$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$
$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$	$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$
$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$	$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$
$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$	$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$
$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$	$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$
$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$	$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$
$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$	$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$
$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$	$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$
$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$	$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$
$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$	$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$

Output: $\mathrm{BWT}(w) = (\qquad\qquad , \;)$

Burrows-Wheeler transform

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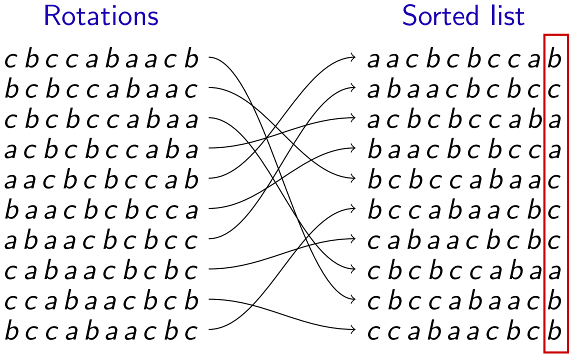

Output: $\mathrm{BWT}(w) = (bcaacccabb,\ \ )$

Burrows-Wheeler transform

- Last column of sorted list of rotations
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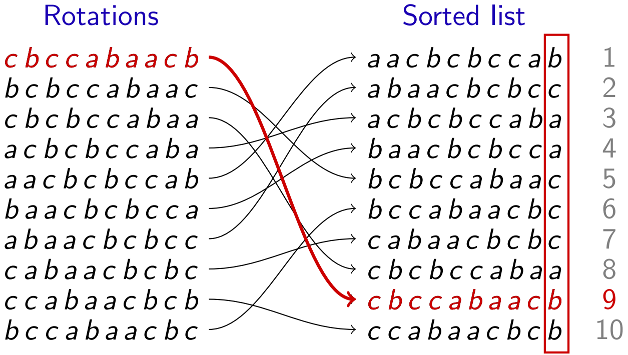

Output: $\mathrm{BWT}(w) = (bcaacccabb, 9)$

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- Last column of sorted list of rotations
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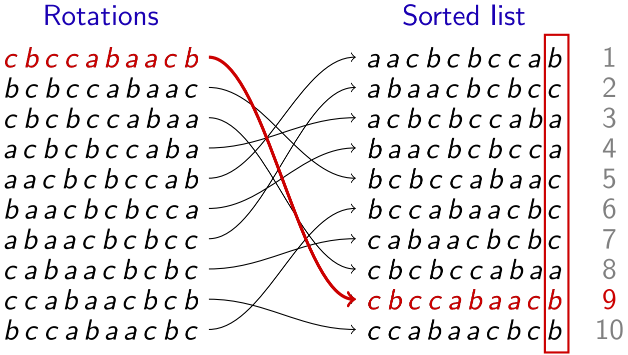
- Example $w = cbccabaacb$



Output: $\mathrm{BWT}(w) = (bcaacccabb, 9)$

- For typical inputs w: $\mathrm{BWT}(w)$ is *easier* to compress

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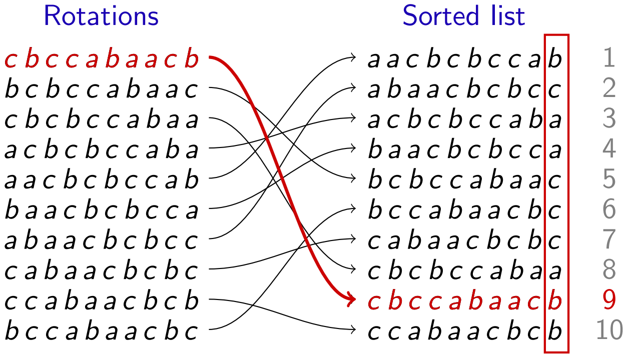

Output: $\text{BWT}(w) = (bcaacccabb, 9)$

- For typical inputs w: $\text{BWT}(w)$ is *easier* to compress
- BWT is not surjective

Inverting the BWT

- BWT(w) $= (bcaacccabb, 9)$

Inverting the BWT

- $\mathrm{BWT}(w) = (bcaacccabb, 9)$

$$\begin{matrix} b \\ c \\ a \\ a \\ c \\ c \\ c \\ a \\ b \\ b \end{matrix}$$

Inverting the BWT

- $\text{BWT}(w) = (bcaacccabb, 9)$

$$\begin{array}{c|} b \\ c \\ a \\ a \\ c \\ c \\ c \\ a \\ b \\ b \end{array}$$

Inverting the BWT

- $\text{BWT}(w) = (bcaacccabb, 9)$

b	a
c	a
a	a
a	b
c	b
c	b
c	c
a	c
b	c
b	c

Inverting the BWT

- $\text{BWT}(w) = (bcaacccabb, 9)$

b	a	a
c	a	b
a	a	c
a	b	a
c	b	c
c	b	c
c	c	a
a	c	b
b	c	b
b	c	c

Inverting the BWT

- $\mathrm{BWT}(w) = (bcaacccabb, 9)$

$$
\begin{array}{c|ccc}
\color{red}{b} & a & a & c \\
\color{red}{c} & a & b & a \\
\color{red}{a} & a & c & b \\
\color{red}{a} & b & a & a \\
\color{red}{c} & b & c & b \\
\color{red}{c} & b & c & c \\
\color{red}{c} & c & a & b \\
\color{red}{a} & c & b & c \\
\color{red}{b} & c & b & c \\
\color{red}{b} & c & c & a
\end{array}
$$

Inverting the BWT

- $\mathrm{BWT}(w) = (bcaacccabb, 9)$

Sorted list

b	$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
c	$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$
a	$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$
a	$b\ a\ a\ c\ b\ c\ b\ c\ c\ a$
c	$b\ c\ b\ c\ c\ a\ b\ a\ a\ c$
c	$b\ c\ c\ a\ b\ a\ a\ c\ b\ c$
c	$c\ a\ b\ a\ a\ c\ b\ c\ b\ c$
a	$c\ b\ c\ b\ c\ c\ a\ b\ a\ a$
b	$c\ b\ c\ c\ a\ b\ a\ a\ c\ b$
b	$c\ c\ a\ b\ a\ a\ c\ b\ c\ b$

Inverting the BWT

- $\text{BWT}(w) = (bcaacccabb, 9)$

Sorted list

b	a	a	c	b	c	b	c	c	a	b	
c	a	b	a	a	c	b	c	b	c	c	
a	a	c	b	c	b	c	c	a	b	a	
a	b	a	a	c	b	c	b	c	c	a	
c	b	c	b	c	c	a	b	a	a	c	
c	b	c	c	a	b	a	a	c	b	c	
c	c	a	b	a	a	c	b	c	b	c	
a	c	b	c	b	c	c	a	b	a	a	
b	c	b	c	c	a	b	a	a	c	b	w
b	c	c	a	b	a	a	c	b	c	b	

Inverting the BWT

- BWT(w) $= (bcaacccabb, 9)$

Sorted list

b	$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$	
c	$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$	
a	$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$	
a	$b\ a\ a\ c\ b\ c\ b\ c\ c\ a$	
c	$b\ c\ b\ c\ c\ a\ b\ a\ a\ c$	
c	$b\ c\ c\ a\ b\ a\ a\ c\ b\ c$	
c	$c\ a\ b\ a\ a\ c\ b\ c\ b\ c$	
a	$c\ b\ c\ b\ c\ c\ a\ b\ a\ a$	
b	$c\ b\ c\ c\ a\ b\ a\ a\ c\ b$	w
b	$c\ c\ a\ b\ a\ a\ c\ b\ c\ b$	

- BWT is injective

The bijective BWT

- Lyndon factorization followd by multi-word BWT

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- Sort order $\preceq$: $u \preceq v$ if $u^\omega \leq v^\omega$

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Rotations

$a\,a\,c\,b$
$b\,a\,a\,c$
$c\,b\,a\,a$
$a\,c\,b\,a$

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Rotations

$a\,a\,c\,b$
$b\,a\,a\,c$
$c\,b\,a\,a$
$a\,c\,b\,a$
$a\,b$
$b\,a$

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Rotations

$a\,a\,c\,b$
$b\,a\,a\,c$
$c\,b\,a\,a$
$a\,c\,b\,a$
$a\,b$
$b\,a$
$b\,c\,c$
$c\,b\,c$
$c\,c\,b$
c

The bijective BWT

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- Example: $w = cbccabaacb$
- Lyndon factorization: $w = c \cdot bcc \cdot ab \cdot aacb$
- Sort order $\preceq$: $u \preceq v$ if $u^\omega \leq v^\omega$

Rotations		Sorted list
$a\,a\,c\,b$	$\longrightarrow$	$a\,a\,c\,b$
$b\,a\,a\,c$		
$c\,b\,a\,a$		
$a\,c\,b\,a$		
$a\,b$		
$b\,a$		
$b\,c\,c$		
$c\,b\,c$		
$c\,c\,b$		
c		

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Rotations	Sorted list
$a\,a\,c\,b$	$a\,a\,c\,b$
$b\,a\,a\,c$	$a\,b$
$c\,b\,a\,a$	
$a\,c\,b\,a$	
$a\,b$	
$b\,a$	
$b\,c\,c$	
$c\,b\,c$	
$c\,c\,b$	
c	

The bijective BWT

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Rotations	Sorted list
$a\,a\,c\,b$	$a\,a\,c\,b$
$b\,a\,a\,c$	$a\,b$
$c\,b\,a\,a$	$a\,c\,b\,a$
$a\,c\,b\,a$	
$a\,b$	
$b\,a$	
$b\,c\,c$	
$c\,b\,c$	
$c\,c\,b$	
c	

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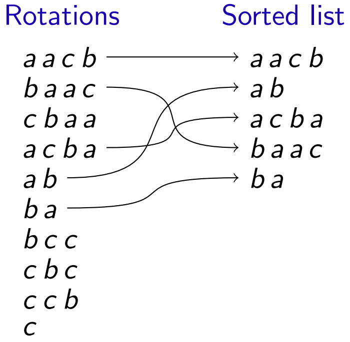

The bijective BWT

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- Example: $w = cbccabaacb$
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Rotations	Sorted list
$a\,a\,c\,b$	$a\,a\,c\,b$
$b\,a\,a\,c$	$a\,b$
$c\,b\,a\,a$	$a\,c\,b\,a$
$a\,c\,b\,a$	$b\,a\,a\,c$
$a\,b$	$b\,a$
$b\,a$	$b\,c\,c$
$b\,c\,c$	$c\,b\,a\,a$
$c\,b\,c$	$c\,b\,c$
$c\,c\,b$	$c\,c\,b$
c	c

The bijective BWT

- Lyndon factorization followd by multi-word BWT
- Example: $w = cbccabaacb$
- Lyndon factorization: $w = c \cdot bcc \cdot ab \cdot aacb$
- Sort order $\preceq$: $u \preceq v$ if $u^\omega \leq v^\omega$

Rotations	Sorted list	Last
$a\,a\,c\,b$	$a\,a\,c\,b$	b
$b\,a\,a\,c$	$a\,b$	b
$c\,b\,a\,a$	$a\,c\,b\,a$	a
$a\,c\,b\,a$	$b\,a\,a\,c$	c
$a\,b$	$b\,a$	a
$b\,a$	$b\,c\,c$	c
$b\,c\,c$	$c\,b\,a\,a$	a
$c\,b\,c$	$c\,b\,c$	c
$c\,c\,b$	$c\,c\,b$	b
c	c	c

The bijective BWT

- Lyndon factorization followd by multi-word BWT
- Example: $w = cbccabaacb$
- Lyndon factorization: $w = c \cdot bcc \cdot ab \cdot aacb$
- Sort order $\preceq$: $u \preceq v$ if $u^\omega \leq v^\omega$

Rotations	Sorted list	Last
$a\,a\,c\,b$	$a\,a\,c\,b$	b
$b\,a\,a\,c$	$a\,b$	b
$c\,b\,a\,a$	$a\,c\,b\,a$	a
$a\,c\,b\,a$	$b\,a\,a\,c$	c
$a\,b$	$b\,a$	a
$b\,a$	$b\,c\,c$	c
$b\,c\,c$	$c\,b\,a\,a$	a
$c\,b\,c$	$c\,b\,c$	c
$c\,c\,b$	$c\,c\,b$	b
c	c	c

- $\mathrm{BWTS}(w) = bbacacacbc$, no index required

Inverting the BWTS

- $\text{BWTS}(w) = bbacacacbc$

Inverting the BWTS

- BWTS(w) = *bbacacacbc*

$$\begin{array}{c|}
b \\
b \\
a \\
c \\
a \\
c \\
a \\
c \\
b \\
c
\end{array}$$

Inverting the BWTS

- BWTS(w) = $bbacacacbc$

b	a
b	a
a	a
c	b
a	b
c	b
a	c
c	c
b	c
c	c

Inverting the BWTS

- BWTS(w) = $bbacacacbc$

b	a	a
b	a	b
a	a	c
c	b	a
a	b	a
c	b	c
a	c	b
c	c	b
b	c	c
c	c	c

Inverting the BWTS

- BWTS(w) $=$ $bbacacacbc$

$$
\begin{array}{c|ccc}
b & a & a & c \\
b & a & b & a \\
a & a & c & b \\
c & b & a & a \\
a & b & a & b \\
c & b & c & c \\
a & c & b & a \\
c & c & b & c \\
b & c & c & b \\
c & c & c & c
\end{array}
$$

Inverting the BWTS

- $\mathrm{BWTS}(w) = bbacacacbc$

Sorted list

b	$a\ a\ c\ b\ a\ a\ c\ b\ a\ a$
b	$a\ b\ a\ b\ a\ b\ a\ b\ a\ b$
a	$a\ c\ b\ a\ a\ c\ b\ a\ a\ c$
c	$b\ a\ a\ c\ b\ a\ a\ c\ b\ a$
a	$b\ a\ b\ a\ b\ a\ b\ a\ b\ a$
c	$b\ c\ c\ b\ c\ c\ b\ c\ c\ b$
a	$c\ b\ a\ a\ c\ b\ a\ a\ c\ b$
c	$c\ b\ c\ c\ b\ c\ c\ b\ c\ c$
b	$c\ c\ b\ c\ c\ b\ c\ c\ b\ c$
c	$c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

Inverting the BWTS

- $\text{BWTS}(w) = bbacacacbc$

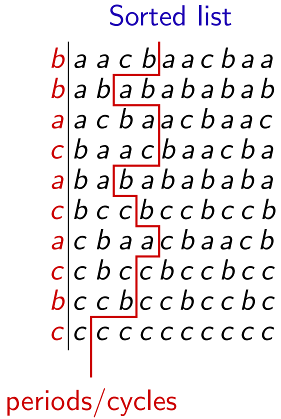
Sorted list

b	*a*	*a*	*c*	*b*	*a*	*a*	*c*	*b*	*a*	*a*
b	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*
a	*a*	*c*	*b*	*a*	*a*	*c*	*b*	*a*	*a*	*c*
c	*b*	*a*	*a*	*c*	*b*	*a*	*a*	*c*	*b*	*a*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*b*	*c*	*c*	*b*	*c*	*c*	*b*	*c*	*c*	*b*
a	*c*	*b*	*a*	*a*	*c*	*b*	*a*	*a*	*c*	*b*
c	*c*	*b*	*c*	*c*	*b*	*c*	*c*	*b*	*c*	*c*
b	*c*	*c*	*b*	*c*	*c*	*b*	*c*	*c*	*b*	*c*
c	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*

periods/cycles

Inverting the BWTS

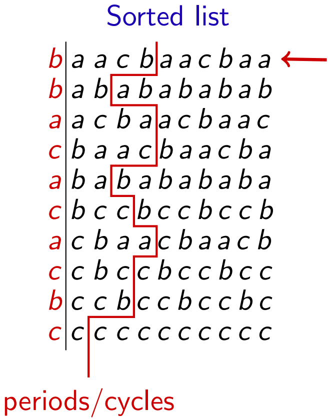
- BWTS(w) = $bbacacacbc$



- $w =$

Inverting the BWTS

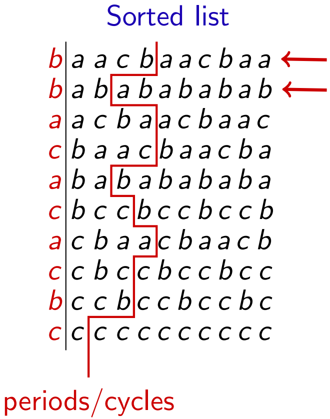
- BWTS(w) $= bbacacacbc$



- $w =$ $aacb$

Inverting the BWTS

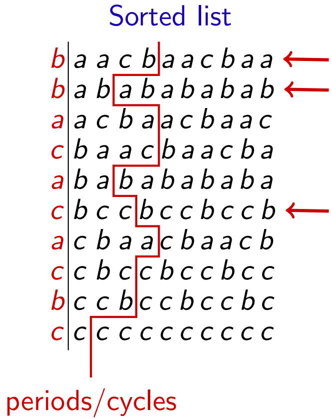
- BWTS(w) $= bbacacacbc$



- $w =$ $ab \cdot aacb$

Inverting the BWTS

- $\mathrm{BWTS}(w) = bbacacacbc$



- $w = \quad bcc \cdot ab \cdot aacb$

Inverting the BWTS

- $\text{BWTS}(w) = bbacacacbc$

Sorted list

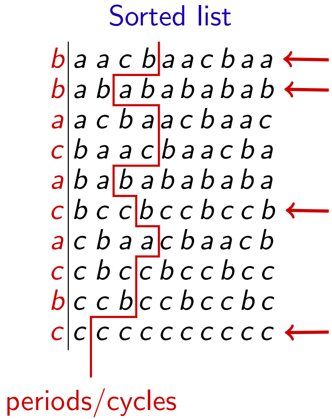
b	$a\,a\,c\,b\,a\,a\,c\,b\,a\,a$
b	$a\,b\,a\,b\,a\,b\,a\,b\,a\,b$
a	$a\,c\,b\,a\,a\,c\,b\,a\,a\,c$
c	$b\,a\,a\,c\,b\,a\,a\,c\,b\,a$
a	$b\,a\,b\,a\,b\,a\,b\,a\,b\,a$
c	$b\,c\,c\,b\,c\,c\,b\,c\,c\,b$
a	$c\,b\,a\,a\,c\,b\,a\,a\,c\,b$
c	$c\,b\,c\,c\,b\,c\,c\,b\,c\,c$
b	$c\,c\,b\,c\,c\,b\,c\,c\,b\,c$
c	$c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

periods/cycles

- $w = c \cdot bcc \cdot ab \cdot aacb$

Inverting the BWTS

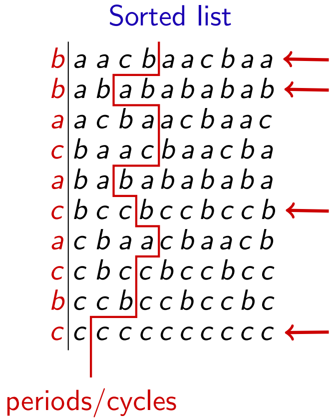
- $\text{BWTS}(w) = bbacacacbc$



- $w = c \cdot bcc \cdot ab \cdot aacb$
- BWTS is bijective

Inverting the BWTS

- BWTS(w) $=$ *bbacacacbc*



- $w = c \cdot bcc \cdot ab \cdot aacb$
- BWTS is bijective
- In fact, BWTS recovers the Lyndon factorization.

Historical remarks about the bijective BWT

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- Scott (2007):
  apply multi-word BWT to blocks obtained by Lyndon factorization, no indices necessary
  (example, computer program, but no formal proof)

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- Gil and Scott (2009, indepently of this paper):
  full description / formal proof of the bijective BWT, test results

Sort transform

- Discovered by Schindler (1997)

Sort transform

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  fast algorithms for the inverse

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Sort transform

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Rotations

$c\ b\ c\ c\ a\ b\ a\ a\ c\ b$
$b\ c\ b\ c\ c\ a\ b\ a\ a\ c$
$c\ b\ c\ b\ c\ c\ a\ b\ a\ a$
$a\ c\ b\ c\ b\ c\ c\ a\ b\ a$
$a\ a\ c\ b\ c\ b\ c\ c\ a\ b$
$b\ a\ a\ c\ b\ c\ b\ c\ c\ a$
$a\ b\ a\ a\ c\ b\ c\ b\ c\ c$
$c\ a\ b\ a\ a\ c\ b\ c\ b\ c$
$c\ c\ a\ b\ a\ a\ c\ b\ c\ b$
$b\ c\ c\ a\ b\ a\ a\ c\ b\ c$

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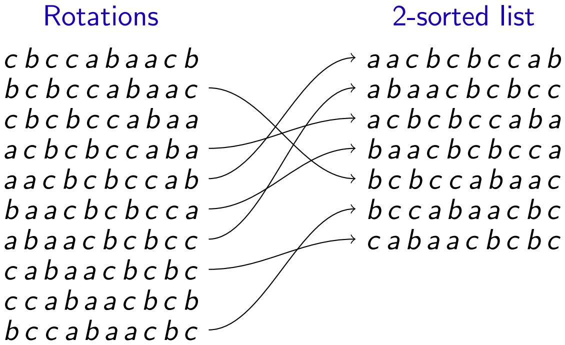

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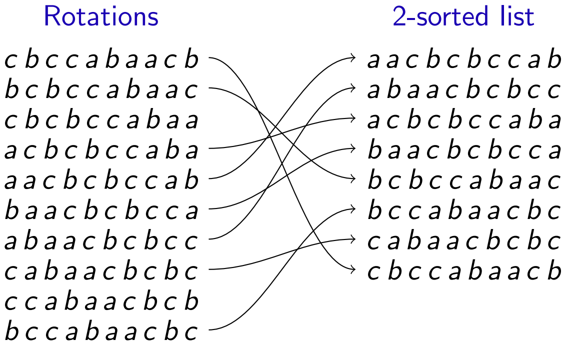

Sort transform

- Discovered by Schindler (1997)
- Nong, Zhang, and Chan (2006/2007/2008):
  fast algorithms for the inverse
- Example $w = cbccabaacb$, order 2

Rotations	2-sorted list
$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$	$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$
$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$	$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$
$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$	$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$
$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$	$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$
$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$	$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$
$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$	$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$
$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$	$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$
$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$	$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$
$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$	$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$
$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$	$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$

Sort transform

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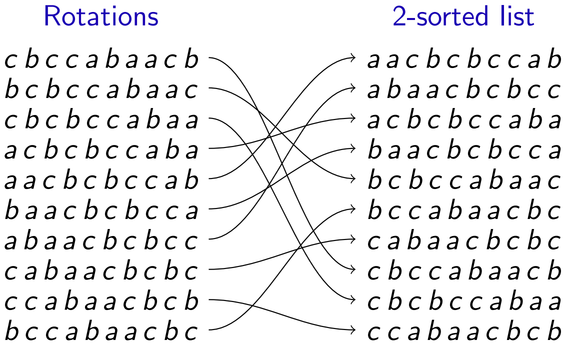


Output: $\mathrm{ST}_2(w) = (\qquad\qquad\quad , \ )$

Sort transform

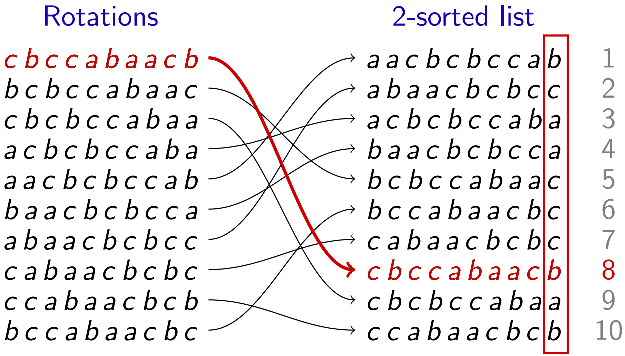
- Discovered by Schindler (1997)
- Nong, Zhang, and Chan (2006/2007/2008):
  fast algorithms for the inverse
- Example $w = cbccabaacb$, order 2

Rotations	2-sorted list
$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$	$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$
$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$	$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$
$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$	$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$
$a\,c\,b\,c\,b\,c\,c\,a\,b\,a$	$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$
$a\,a\,c\,b\,c\,b\,c\,c\,a\,b$	$b\,c\,b\,c\,c\,a\,b\,a\,a\,c$
$b\,a\,a\,c\,b\,c\,b\,c\,c\,a$	$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$
$a\,b\,a\,a\,c\,b\,c\,b\,c\,c$	$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$
$c\,a\,b\,a\,a\,c\,b\,c\,b\,c$	$c\,b\,c\,c\,a\,b\,a\,a\,c\,b$
$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$	$c\,b\,c\,b\,c\,c\,a\,b\,a\,a$
$b\,c\,c\,a\,b\,a\,a\,c\,b\,c$	$c\,c\,a\,b\,a\,a\,c\,b\,c\,b$

Output: $\mathrm{ST}_2(w) = (bcaacccbab,\ \ )$

Sort transform

- Discovered by Schindler (1997)
- Nong, Zhang, and Chan (2006/2007/2008):
  fast algorithms for the inverse
- Example $w = cbccabaacb$, order 2



Output: $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

b
c
a
a
c
c
c
b
a
b

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

b	a
c	a
a	a
a	b
c	b
c	b
c	c
b	c
a	c
b	c

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	$a\ a$
c	$a\ b$
a	$a\ c$
a	$b\ a$
c	$b\ c$
c	$b\ c$
c	$c\ a$
b	$c\ b$
a	$c\ b$
b	$c\ c$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	$a\ a$
c	$a\ b$
a	$a\ c$
a	$b\ a$
c	$b\ c$
c	$b\ c$
c	$c\ a$
b	$c\ b$
a	$c\ b$
b	$c\ c$

$w =$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	a	a
c	a	b
a	a	c
a	b	a
c	b	c
c	b	c
c	c	a
b	c	b ⟵ *start*
a	c	b
b	c	c

$w = \qquad\qquad \mid cb$

Inverting the ST

- $ST_2(w) = (bcaacccbab, 8)$

2-contexts

b	$a\ a$
c	$a\ b$
a	$a\ c$
a	$b\ a$
c	$b\ c$
c	$b\ c$
c	$c\ a$
~~b~~	~~$c\ b$~~ ⟵ *start*
a	$c\ b$
b	$c\ c$

$w = \quad b \mid cb$

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	$a\ a$
c	$a\ b$
a	$a\ c$
a	$b\ a$
~~c~~	~~$b\ c$~~
c	$b\ c$
c	$c\ a$
~~b~~	~~$c\ b$~~ ← *start*
a	$c\ b$
b	$c\ c$

$w = \quad cb \mid cb$

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	*a*	*a*
c	*a*	*b*
a	*a*	*c*
a	*b*	*a*
~~*c*~~	~~*b*~~	~~*c*~~
c	*b*	*c*
c	*c*	*a*
~~*b*~~	~~*c*~~	~~*b*~~ ← *start*
~~*a*~~	~~*c*~~	~~*b*~~
b	*c*	*c*

$w = \quad acb \mid cb$

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	$a\ a$	
c	$a\ b$	
~~a~~	~~$a\ c$~~	
a	$b\ a$	
~~c~~	~~$b\ c$~~	
c	$b\ c$	
c	$c\ a$	
~~b~~	~~$c\ b$~~	⟵ *start*
~~a~~	~~$c\ b$~~	
b	$c\ c$	

$w = \quad aacb \mid cb$

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

~~b~~	~~a~~	~~a~~
c	a	b
~~a~~	~~a~~	~~c~~
a	b	a
~~c~~	~~b~~	~~c~~
c	b	c
c	c	a
~~b~~	~~c~~	~~b~~ ⟵ *start*
~~a~~	~~c~~	~~b~~
b	c	c

$w = \quad baacb \mid cb$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

~~b~~	~~$a\ a$~~	
c	$a\ b$	
~~a~~	~~$a\ c$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	
c	$b\ c$	
c	$c\ a$	
~~b~~	~~$c\ b$~~	⟵ *start*
~~a~~	~~$c\ b$~~	
b	$c\ c$	

$w = \qquad abaacb \mid cb$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

~~b~~	~~$a\ a$~~	
~~c~~	~~$a\ b$~~	
~~a~~	~~$a\ c$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	
c	$b\ c$	
c	$c\ a$	
~~b~~	~~$c\ b$~~	$\longleftarrow$ *start*
~~a~~	~~$c\ b$~~	
b	$c\ c$	

$w = \quad cabaacb \mid cb$

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

~~b~~	~~$a\ a$~~	
~~c~~	~~$a\ b$~~	
~~a~~	~~$a\ c$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	
c	$b\ c$	
~~c~~	~~$c\ a$~~	
~~b~~	~~$c\ b$~~	⟵ *start*
~~a~~	~~$c\ b$~~	
b	$c\ c$	

$w = \quad ccabaacb \mid cb$

Inverting the ST

- $ST_2(w) = (bcaacccbab, 8)$

2-contexts

~~b~~	~~$a\ a$~~	
~~c~~	~~$a\ b$~~	
~~a~~	~~$a\ c$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	
c	$b\ c$	
~~c~~	~~$c\ a$~~	
~~b~~	~~$c\ b$~~	$\longleftarrow$ *start*
~~a~~	~~$c\ b$~~	
~~b~~	~~$c\ c$~~	

$w = \;\; bccabaacb \mid cb$

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

b	$a\ a$	
c	$a\ b$	
a	$a\ c$	
a	$b\ a$	
c	$b\ c$	
c	$b\ c$	
c	$c\ a$	
b	$c\ b$	⟵ *start*
a	$c\ b$	
b	$c\ c$	

$w = cbccabaacb \mid cb$

Inverting the ST

- $\mathsf{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

~~*b*~~	~~*a a*~~	
~~*c*~~	~~*a b*~~	
~~*a*~~	~~*a c*~~	
~~*a*~~	~~*b a*~~	
~~*c*~~	~~*b c*~~	
~~*c*~~	~~*b c*~~	
~~*c*~~	~~*c a*~~	
~~*b*~~	~~*c b*~~	⟵ *start*
~~*a*~~	~~*c b*~~	
~~*b*~~	~~*c c*~~	

$w = cbccabaacb \mid cb$

- ST is injective

Inverting the ST

- $\mathrm{ST}_2(w) = (bcaacccbab, 8)$

2-contexts

~~b~~	~~$a\ a$~~	
~~c~~	~~$a\ b$~~	
~~a~~	~~$a\ c$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	
~~c~~	~~$b\ c$~~	
~~c~~	~~$c\ a$~~	
~~b~~	~~$c\ b$~~	$\longleftarrow$ *start*
~~a~~	~~$c\ b$~~	
~~b~~	~~$c\ c$~~	

$w = cbccabaacb \mid cb$

- ST is injective
- ST_0 is reversal

The bijective ST

- Lyndon factorization followd by multi-word ST

The bijective ST

- Lyndon factorization followd by multi-word ST
- Example: $w = cbccabaacb$

The bijective ST

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- Lyndon factorization: $w = c \cdot bcc \cdot ab \cdot aacb$

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- Sort order $\preceq$: $u \preceq v$ if $u^\omega \leq v^\omega$

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Rotations

$a\,a\,c\,b$
$b\,a\,a\,c$
$c\,b\,a\,a$
$a\,c\,b\,a$
$a\,b$
$b\,a$
$b\,c\,c$
$c\,b\,c$
$c\,c\,b$
c

The bijective ST

- Lyndon factorization followd by multi-word ST
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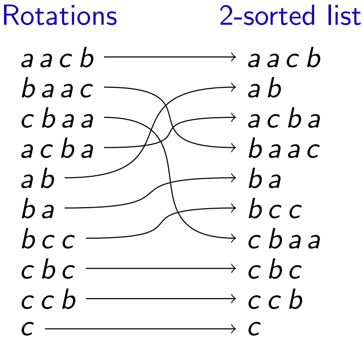

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Rotations	2-sorted list	Last
$a\,a\,c\,b$	$a\,a\,c\,b$	b
$b\,a\,a\,c$	$a\,b$	b
$c\,b\,a\,a$	$a\,c\,b\,a$	a
$a\,c\,b\,a$	$b\,a\,a\,c$	c
$a\,b$	$b\,a$	a
$b\,a$	$b\,c\,c$	c
$b\,c\,c$	$c\,b\,a\,a$	a
$c\,b\,c$	$c\,b\,c$	c
$c\,c\,b$	$c\,c\,b$	b
c	c	c

The bijective ST

- Lyndon factorization followd by multi-word ST
- Example: $w = cbccabaacb$
- Lyndon factorization: $w = c \cdot bcc \cdot ab \cdot aacb$
- Sort order $\preceq$: $u \preceq v$ if $u^\omega \leq v^\omega$

Rotations	2-sorted list	Last
$a\,a\,c\,b$	$a\,a\,c\,b$	b
$b\,a\,a\,c$	$a\,b$	b
$c\,b\,a\,a$	$a\,c\,b\,a$	a
$a\,c\,b\,a$	$b\,a\,a\,c$	c
$a\,b$	$b\,a$	a
$b\,a$	$b\,c\,c$	c
$b\,c\,c$	$c\,b\,a\,a$	a
$c\,b\,c$	$c\,b\,c$	c
$c\,c\,b$	$c\,c\,b$	b
c	c	c

- $\mathrm{LST}_2(w) = bbacacacbc$, no index required

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

$$\begin{array}{c|}
b \\ b \\ a \\ c \\ a \\ c \\ a \\ c \\ b \\ c
\end{array}$$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

b	a
b	a
a	a
c	b
a	b
c	b
a	c
c	c
b	c
c	c

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

b	$a\ a$
b	$a\ b$
a	$a\ c$
c	$b\ a$
a	$b\ a$
c	$b\ c$
a	$c\ b$
c	$c\ b$
b	$c\ c$
c	$c\ c$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

b	$a\ a$
b	$a\ b$
a	$a\ c$
c	$b\ a$
a	$b\ a$
c	$b\ c$
a	$c\ b$
c	$c\ b$
b	$c\ c$
c	$c\ c$

$w =$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

b	$a\ a$ ⟵ *start*
b	$a\ b$
a	$a\ c$
c	$b\ a$
a	$b\ a$
c	$b\ c$
a	$c\ b$
c	$c\ b$
b	$c\ c$
c	$c\ c$

$w =$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~ ⟵ *start*
b	$a\ b$
a	$a\ c$
c	$b\ a$
a	$b\ a$
c	$b\ c$
a	$c\ b$
c	$c\ b$
b	$c\ c$
c	$c\ c$

$w = \qquad b$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~ ← *start*
b	$a\ b$
a	$a\ c$
~~c~~	~~$b\ a$~~
a	$b\ a$
c	$b\ c$
a	$c\ b$
c	$c\ b$
b	$c\ c$
c	$c\ c$

$w = \qquad cb$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	← *start*
b	$a\ b$	
a	$a\ c$	
~~c~~	~~$b\ a$~~	
a	$b\ a$	
c	$b\ c$	
~~a~~	~~$c\ b$~~	
c	$c\ b$	
b	$c\ c$	
c	$c\ c$	

$w =$ acb

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	← *start*
b	$a\ b$	
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
a	$b\ a$	
c	$b\ c$	
~~a~~	~~$c\ b$~~	
c	$c\ b$	
b	$c\ c$	
c	$c\ c$	

$w = \qquad aacb$

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~$b \mid a\ a$~~ ⟵ *start*
$b \mid a\ b$ ⟵ *start* 2
~~$a \mid a\ c$~~
~~$c \mid b\ a$~~
$a \mid b\ a$
$c \mid b\ c$
~~$a \mid c\ b$~~
$c \mid c\ b$
$b \mid c\ c$
$c \mid c\ c$

$w = \qquad\qquad \cdot\, aacb$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	← *start*
~~b~~	~~$a\ b$~~	← *start* 2
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
a	$b\ a$	
c	$b\ c$	
~~a~~	~~$c\ b$~~	
c	$c\ b$	
b	$c\ c$	
c	$c\ c$	

$w = \qquad b \cdot aacb$

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	← *start*
~~b~~	~~$a\ b$~~	← *start* 2
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
~~a~~	~~$b\ a$~~	
c	$b\ c$	
~~a~~	~~$c\ b$~~	
c	$c\ b$	
b	$c\ c$	
c	$c\ c$	

$w = \qquad ab \cdot aacb$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	⟵ *start*
~~b~~	~~$a\ b$~~	⟵ *start* 2
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
~~a~~	~~$b\ a$~~	
c	$b\ c$	⟵ *start* 3
~~a~~	~~$c\ b$~~	
c	$c\ b$	
b	$c\ c$	
c	$c\ c$	

$w = \qquad\qquad \cdot\, ab \cdot aacb$

Inverting the LST

- $\mathsf{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	← *start*
~~b~~	~~$a\ b$~~	← *start* 2
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	← *start* 3
~~a~~	~~$c\ b$~~	
c	$c\ b$	
b	$c\ c$	
c	$c\ c$	

$w = \quad c \cdot ab \cdot aacb$

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	← *start*
~~b~~	~~$a\ b$~~	← *start* 2
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	← *start* 3
~~a~~	~~$c\ b$~~	
~~c~~	~~$c\ b$~~	
b	$c\ c$	
c	$c\ c$	

$w = \quad cc \cdot ab \cdot aacb$

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~*b* | *a a*~~	←	*start*
~~*b* | *a b*~~	←	*start* 2
~~*a* | *a c*~~		
~~*c* | *b a*~~		
~~*a* | *b a*~~		
~~*c* | *b c*~~	←	*start* 3
~~*a* | *c b*~~		
~~*c* | *c b*~~		
~~*b* | *c c*~~		
c | *c c*		

$w = \quad bcc \cdot ab \cdot aacb$

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~*b*~~	~~*a a*~~	← *start*
~~*b*~~	~~*a b*~~	← *start* 2
~~*a*~~	~~*a c*~~	
~~*c*~~	~~*b a*~~	
~~*a*~~	~~*b a*~~	
~~*c*~~	~~*b c*~~	← *start* 3
~~*a*~~	~~*c b*~~	
~~*c*~~	~~*c b*~~	
~~*b*~~	~~*c c*~~	
~~*c*~~	~~*c c*~~	← *start* 4

$w = c \cdot bcc \cdot ab \cdot aacb$

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~b~~	~~$a\ a$~~	⟵ *start*
~~b~~	~~$a\ b$~~	⟵ *start* 2
~~a~~	~~$a\ c$~~	
~~c~~	~~$b\ a$~~	
~~a~~	~~$b\ a$~~	
~~c~~	~~$b\ c$~~	⟵ *start* 3
~~a~~	~~$c\ b$~~	
~~c~~	~~$c\ b$~~	
~~b~~	~~$c\ c$~~	
~~c~~	~~$c\ c$~~	⟵ *start* 4

$w = c \cdot bcc \cdot ab \cdot aacb$

- In general, LST does not recover Lyndon factorization.

Inverting the LST

- $\mathrm{LST}_2(w) = bbacacacbc$

2-contexts

~~*b*~~	~~*a a*~~	← *start*
~~*b*~~	~~*a b*~~	← *start* 2
~~*a*~~	~~*a c*~~	
~~*c*~~	~~*b a*~~	
~~*a*~~	~~*b a*~~	
~~*c*~~	~~*b c*~~	← *start* 3
~~*a*~~	~~*c b*~~	
~~*c*~~	~~*c b*~~	
~~*b*~~	~~*c c*~~	
~~*c*~~	~~*c c*~~	← *start* 4

$w = c \cdot bcc \cdot ab \cdot aacb$

- In general, LST does not recover Lyndon factorization.
- LST_0 is reversal

Comparison of BWTS and LST

- $w = cbccabaacb$

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- $\mathsf{BWTS}(w) = \mathsf{LST}_2(w) = b \cdot b \cdot acacacbc$

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations

$a\,c\,a\,c\,a\,c\,b\,c$
$c\,a\,c\,a\,c\,a\,c\,b$
$b\,c\,a\,c\,a\,c\,a\,c$
$c\,b\,c\,a\,c\,a\,c\,a$
$a\,c\,b\,c\,a\,c\,a\,c$
$c\,a\,c\,b\,c\,a\,c\,a$
$a\,c\,a\,c\,b\,c\,a\,c$
$c\,a\,c\,a\,c\,b\,c\,a$
b
b

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\text{BWTS}(w) = \text{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted
$a\,c\,a\,c\,a\,c\,b\,c$	$a\,c\,a\,c\,a\,c\,b\,c$
$c\,a\,c\,a\,c\,a\,c\,b$	$a\,c\,a\,c\,b\,c\,a\,c$
$b\,c\,a\,c\,a\,c\,a\,c$	$a\,c\,b\,c\,a\,c\,a\,c$
$c\,b\,c\,a\,c\,a\,c\,a$	b
$a\,c\,b\,c\,a\,c\,a\,c$	b
$c\,a\,c\,b\,c\,a\,c\,a$	$b\,c\,a\,c\,a\,c\,a\,c$
$a\,c\,a\,c\,b\,c\,a\,c$	$c\,a\,c\,a\,c\,a\,c\,b$
$c\,a\,c\,a\,c\,b\,c\,a$	$c\,a\,c\,a\,c\,b\,c\,a$
b	$c\,a\,c\,b\,c\,a\,c\,a$
b	$c\,b\,c\,a\,c\,a\,c\,a$

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS
$a\,c\,a\,c\,a\,c\,b\,c$	$a\,c\,a\,c\,a\,c\,b\,c$	c
$c\,a\,c\,a\,c\,a\,c\,b$	$a\,c\,a\,c\,b\,c\,a\,c$	c
$b\,c\,a\,c\,a\,c\,a\,c$	$a\,c\,b\,c\,a\,c\,a\,c$	c
$c\,b\,c\,a\,c\,a\,c\,a$	b	b
$a\,c\,b\,c\,a\,c\,a\,c$	b	b
$c\,a\,c\,b\,c\,a\,c\,a$	$b\,c\,a\,c\,a\,c\,a\,c$	c
$a\,c\,a\,c\,b\,c\,a\,c$	$c\,a\,c\,a\,c\,a\,c\,b$	b
$c\,a\,c\,a\,c\,b\,c\,a$	$c\,a\,c\,a\,c\,b\,c\,a$	a
b	$c\,a\,c\,b\,c\,a\,c\,a$	a
b	$c\,b\,c\,a\,c\,a\,c\,a$	a

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS	1-sorted
a c a c a c b c	*a c a c a c b c*	*c*	*a c a c a c b c*
c a c a c a c b	*a c a c b c a c*	*c*	*a c b c a c a c*
b c a c a c a c	*a c b c a c a c*	*c*	*a c a c b c a c*
c b c a c a c a	*b*	*b*	*b c a c a c a c*
a c b c a c a c	*b*	*b*	*b*
c a c b c a c a	*b c a c a c a c*	*c*	*b*
a c a c b c a c	*c a c a c a c b*	*b*	*c a c a c a c b*
c a c a c b c a	*c a c a c b c a*	*a*	*c b c a c a c a*
b	*c a c b c a c a*	*a*	*c a c b c a c a*
b	*c b c a c a c a*	*a*	*c a c a c b c a*

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS	1-sorted	ST_1
a c a c a c b c	*a c a c a c b c*	*c*	*a c a c a c b c*	*c*
c a c a c a c b	*a c a c b c a c*	*c*	*a c b c a c a c*	*c*
b c a c a c a c	*a c b c a c a c*	*c*	*a c a c b c a c*	*c*
c b c a c a c a	*b*	*b*	*b c a c a c a c*	*c*
a c b c a c a c	*b*	*b*	*b*	*b*
c a c b c a c a	*b c a c a c a c*	*c*	*b*	*b*
a c a c b c a c	*c a c a c a c b*	*b*	*c a c a c a c b*	*b*
c a c a c b c a	*c a c a c b c a*	*a*	*c b c a c a c a*	*a*
b	*c a c b c a c a*	*a*	*c a c b c a c a*	*a*
b	*c b c a c a c a*	*a*	*c a c a c b c a*	*a*

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS	1-sorted	ST_1
$a\,c\,a\,c\,a\,c\,b\,c$	$a\,c\,a\,c\,a\,c\,b\,c$	c	$a\,c\,a\,c\,a\,c\,b\,c$	c
$c\,a\,c\,a\,c\,a\,c\,b$	$a\,c\,a\,c\,b\,c\,a\,c$	c	$a\,c\,b\,c\,a\,c\,a\,c$	c
$b\,c\,a\,c\,a\,c\,a\,c$	$a\,c\,b\,c\,a\,c\,a\,c$	c	$a\,c\,a\,c\,b\,c\,a\,c$	c
$c\,b\,c\,a\,c\,a\,c\,a$	b	b	$b\,c\,a\,c\,a\,c\,a\,c$	c
$a\,c\,b\,c\,a\,c\,a\,c$	b	b	b	b
$c\,a\,c\,b\,c\,a\,c\,a$	$b\,c\,a\,c\,a\,c\,a\,c$	c	b	b
$a\,c\,a\,c\,b\,c\,a\,c$	$c\,a\,c\,a\,c\,a\,c\,b$	b	$c\,a\,c\,a\,c\,a\,c\,b$	b
$c\,a\,c\,a\,c\,b\,c\,a$	$c\,a\,c\,a\,c\,b\,c\,a$	a	$c\,b\,c\,a\,c\,a\,c\,a$	a
b	$c\,a\,c\,b\,c\,a\,c\,a$	a	$c\,a\,c\,b\,c\,a\,c\,a$	a
b	$c\,b\,c\,a\,c\,a\,c\,a$	a	$c\,a\,c\,a\,c\,b\,c\,a$	a

- $\mathrm{BWTS}\big(\mathrm{BWTS}(w)\big) = cccbbcbaaa$

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS	1-sorted	ST_1
$a\,c\,a\,c\,a\,c\,b\,c$	$a\,c\,a\,c\,a\,c\,b\,c$	c	$a\,c\,a\,c\,a\,c\,b\,c$	c
$c\,a\,c\,a\,c\,a\,c\,b$	$a\,c\,a\,c\,b\,c\,a\,c$	c	$a\,c\,b\,c\,a\,c\,a\,c$	c
$b\,c\,a\,c\,a\,c\,a\,c$	$a\,c\,b\,c\,a\,c\,a\,c$	c	$a\,c\,a\,c\,b\,c\,a\,c$	c
$c\,b\,c\,a\,c\,a\,c\,a$	b	b	$b\,c\,a\,c\,a\,c\,a\,c$	c
$a\,c\,b\,c\,a\,c\,a\,c$	b	b	b	b
$c\,a\,c\,b\,c\,a\,c\,a$	$b\,c\,a\,c\,a\,c\,a\,c$	c	b	b
$a\,c\,a\,c\,b\,c\,a\,c$	$c\,a\,c\,a\,c\,a\,c\,b$	b	$c\,a\,c\,a\,c\,a\,c\,b$	b
$c\,a\,c\,a\,c\,b\,c\,a$	$c\,a\,c\,a\,c\,b\,c\,a$	a	$c\,b\,c\,a\,c\,a\,c\,a$	a
b	$c\,a\,c\,b\,c\,a\,c\,a$	a	$c\,a\,c\,b\,c\,a\,c\,a$	a
b	$c\,b\,c\,a\,c\,a\,c\,a$	a	$c\,a\,c\,a\,c\,b\,c\,a$	a

- $\mathrm{BWTS}\big(\mathrm{BWTS}(w)\big) = cccbbcbaaa$
- $\mathrm{ST}_1\big(\mathrm{ST}_2(w)\big) = ccccbbbaaa$

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS	1-sorted	ST_1
$acacacbc$	$acacacbc$	c	$acacacbc$	c
$cacacacb$	$acacbcac$	c	$acbcacac$	c
$bcacacac$	$acbcacac$	c	$acacbcac$	c
$cbcacaca$	b	b	$bcacacac$	c
$acbcacac$	b	b	b	b
$cacbcaca$	$bcacacac$	c	b	b
$acacbcac$	$cacacacb$	b	$cacacacb$	b
$cacacbca$	$cacacbca$	a	$cbcacaca$	a
b	$cacbcaca$	a	$cacbcaca$	a
b	$cbcacaca$	a	$cacacbca$	a

- $\mathrm{BWTS}\big(\mathrm{BWTS}(w)\big) = cccbbcbaaa$
- $\mathrm{ST}_1\big(\mathrm{ST}_2(w)\big) = ccccbbbaaa$
- BWT / BWTS mixes letters within contexts

Comparison of BWTS and LST

- $w = cbccabaacb$
- $\mathrm{BWTS}(w) = \mathrm{LST}_2(w) = b \cdot b \cdot acacacbc$

rotations	sorted	BWTS	1-sorted	ST_1
$a\,c\,a\,c\,a\,c\,b\,c$	$a\,c\,a\,c\,a\,c\,b\,c$	c	$a\,c\,a\,c\,a\,c\,b\,c$	c
$c\,a\,c\,a\,c\,a\,c\,b$	$a\,c\,a\,c\,b\,c\,a\,c$	c	$a\,c\,b\,c\,a\,c\,a\,c$	c
$b\,c\,a\,c\,a\,c\,a\,c$	$a\,c\,b\,c\,a\,c\,a\,c$	c	$a\,c\,a\,c\,b\,c\,a\,c$	c
$c\,b\,c\,a\,c\,a\,c\,a$	b	b	$b\,c\,a\,c\,a\,c\,a\,c$	c
$a\,c\,b\,c\,a\,c\,a\,c$	b	b	b	b
$c\,a\,c\,b\,c\,a\,c\,a$	$b\,c\,a\,c\,a\,c\,a\,c$	c	b	b
$a\,c\,a\,c\,b\,c\,a\,c$	$c\,a\,c\,a\,c\,a\,c\,b$	b	$c\,a\,c\,a\,c\,a\,c\,b$	b
$c\,a\,c\,a\,c\,b\,c\,a$	$c\,a\,c\,a\,c\,b\,c\,a$	a	$c\,b\,c\,a\,c\,a\,c\,a$	a
b	$c\,a\,c\,b\,c\,a\,c\,a$	a	$c\,a\,c\,b\,c\,a\,c\,a$	a
b	$c\,b\,c\,a\,c\,a\,c\,a$	a	$c\,a\,c\,a\,c\,b\,c\,a$	a

- $\mathrm{BWTS}\big(\mathrm{BWTS}(w)\big) = cccbbcbaaa$
- $\mathrm{ST}_1\big(\mathrm{ST}_2(w)\big) = ccccbbbaaa$
- BWT / BWTS mixes letters within contexts
- ST_k / LST_k preserves order within k-contexts

Open problems

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- Algorithmic aspects

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- Algorithmic aspects
- Multiple application of transforms

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- Algorithmic aspects
- Multiple application of transforms
- Different orders for two main steps of BWTS

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- Algorithmic aspects
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- Manzini-type analysis of BWTS / ST / LST

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Thank you!