Reducing Repetitions

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Crossing Over

\[ \delta \gamma \]

\[ \alpha \beta \]

\[ u \]

\[ \gamma \]

\[ \delta \]

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Reducing Repetitions
Duplication

\[
\begin{align*}
\text{u} & \quad \text{u}_1 \quad z \quad \text{u}_2 \\
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\end{align*}
\]
Graphical Display of Duplication

P. Leupold  Reducing Repetitions
We formalize this with the duplication relation $\heartsuit$ defined as

$$u \heartsuit v \iff \exists z [z \in \Sigma^+ \land u = u_1zu_2 \land v = u_1zzu_2].$$
The Duplication Operation

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$$u \heartsuit v :\iff \exists z [z \in \Sigma^+ \land u = u_1zu_2 \land v = u_1zzu_2].$$

Note that the contexts $u_1$ and $u_2$ form part of the relation.
The Duplication Operation

We formalize this with the duplication relation $\diamondsuit$ defined as

$$u \diamondsuit v :\equiv \exists z [z \in \Sigma^+ \land u = u_1zu_2 \land v = u_1zuzu_2].$$

Note that the contexts $u_1$ and $u_2$ form part of the relation.

We also consider variants with length bounds $|z| \leq k$ or $|z| = k$ and we write $\diamondsuit^{\leq k}$ or $\diamondsuit^k$ respectively.
Theorem

\( \cdot \) preserves regularity over two-letter alphabet, but not over three-letter alphabet.

Open Problem

Does \( \cdot \) preserve context-freeness?
Theorem

\( \heartsuit^n \) preserves regularity over two-letter alphabet, but not over three-letter alphabet.

Theorem

\( \heartsuit \leq_k^n \) preserves context-freeness.
Theorem

\( \heartsuit^* \) preserves regularity over two-letter alphabet, but not over three-letter alphabet.

Theorem

\( \heartsuit \leq k^* \) preserves context-freeness.

Open Problem

Does \( \heartsuit^* \) preserve context-freeness?
For reducing repetitions we will use the inverse of ♦ denoted by ⇒.
Duplication Roots

For reducing repetitions we will use the inverse of $\heartsuit$ denoted by $\Rightarrow$.

**Definition**

The *duplication root* of a non-empty word $w$ is

$$\sqrt{w} := \text{IRR}(\Rightarrow) \cap \{ u : w \Rightarrow^* u \}.$$  

As usual, this notion is extended in the canonical way from words to languages such that

$$\sqrt{L} := \bigcup_{w \in L} \sqrt{w}.$$  

The roots $\sqrt[k]{w}$ and $\heartsuit^{k}\sqrt{w}$ are defined in completely analogous ways.
• All words in a duplication root are square-free, and over an alphabet of two letters only the seven square-free words \{\lambda, a, b, ab, ba, aba, bab\} exist. They are uniquely determined by their first letter, the last letter, and the set of letters occurring in them.
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• By undoing duplications, i.e., by applying rules from \(\to\), we obtain from the word \(w = abcbabcbc\) the words in the set \{abc, abcbc, abcabc\}. Thus we have the root \(\sqrt{abcbabcbc} = \{abc, abcabc\}\).
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• By undoing duplications, i.e., by applying rules from \(\rightarrow\), we obtain from the word \(w = abcbabcbc\) the words in the set \(\{abc, abc\). \)

Thus we have the root \(\sqrt{abcbabcbc} = \{abc, abc\). \)

• \(\sqrt{babacabacbcabacb} = \{bacabacb, bacbcabacb, bacb\),

and

\[\sqrt{ababcbabcacbcababcababcababcab} =\]

\[\{abcbabcabacbcab, abcbabcab, abcabcabcab, abcbacababcab, abc\),\]
Theorem

The closure properties of the classes of regular and context-free languages under the three duplication roots are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$\heartsuit^k L$</th>
<th>$\heartsuit^{\leq k} L$</th>
<th>$\heartsuit L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>REG</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>CF</td>
<td>?</td>
<td>?</td>
<td>N</td>
</tr>
</tbody>
</table>

The symbol Y stands for closure, N stands for non-closure, and ? means that the problem is open.
In an effort to define a new measure for the complexity of words, Ilie et al. defined a reduction relation very similar to undoing duplications, which however remembers the steps it takes. For the definition let $D = \{0, 1, \ldots, 9\}$ be the set of decimal digits, and $\Sigma$ be an alphabet disjoint from $D$. The alphabet for the reduction relation is $T := \Sigma \cup D \cup \{\langle, \rangle, EXP\}$.

Then the reduction relation $\Rightarrow$ is defined by $u \Rightarrow v$ iff $u = u_1 x^n u_2$, $v = u_1 \langle x \rangle \exp \langle \text{dec } n \rangle u_2$ for some $u_1, u_2 \in T^*$, $x = \Sigma^+$, $n > 2$. Finally, let $h$ be the morphism erasing all symbols except the letters from $\Sigma$. 
Example

For the word $ababcbc$ there are two irreducible forms under $\Rightarrow$, namely $\langle ab \rangle \text{EXP}\langle 2 \rangle cbc$ and $aba\langle bc \rangle \text{EXP}\langle 2 \rangle$. 
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Under $\Rightarrow$, however, the images of both words under $h$ are further reducible to a common normal form: both $ababcbc \Rightarrow abc$ and $ababcbc \Rightarrow ababc \Rightarrow abc$ are possible reductions leading to $abc$. 
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Notice how the brackets block the further reduction of $abab$ in $aba \langle bc \rangle \ EXP\langle 2 \rangle$ and of $bcbc$ in $\langle ab \rangle \ EXP\langle 2 \rangle cbc$. 
There are two main differences between the two relations.

1. A reduction $u \Rightarrow \langle u \rangle_{\text{EXP}} \langle n \rangle$ is done in a single step while the reduction $u \Rightarrow u^* u$ will always take $n - 1$ steps.

2. If $w \Rightarrow^* u$ then $w \Rightarrow h(u)$, but the reverse does not hold, see the Example above.

Despite these differences, the similarities are evident, and $\Rightarrow^*$ can be embedded in $\Rightarrow^*$. We state a further relation.

Theorem

For a word $w$, if $\sqrt[w]{} \subseteq \{h(u) : w \Rightarrow^* u\}$ then $|\sqrt[w]{}| = 1$.
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Unduplication versus Repetition Complexity

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For a word $w$, if $\sqrt[n]{w} \subseteq \{ h(u) : w \Rightarrow^* u \}$ then $|\sqrt[n]{w}| = 1$. 

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Reducing Repetitions
The Number of Duplication Roots

We have seen from the examples above that the number of possible duplication roots seems to increase with increasing word length. Our main interest here is to investigate the behaviour of the function:

\[ \text{duproots}(n) := \max\{|\sqrt[n]{w}| : |w| = n\}. \]
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Because it has often turned out to be very useful to consider problems about duplications with a length restriction, we also define the function

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Notice that we do not bound the alphabet size.
Obviously, rules from \( \Rightarrow \) can only be applied on square factors. Thus the number of squares is the number of possible distinct rule applications in a string.
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**Fact**

Let $w$ be a word with period $k$. Then all applications of rules from $\Rightarrow_k$ will result in the same word, i.e. $\{u : w \Rightarrow_k u\}$ is a singleton set.
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**Fact**

Let $w$ be a word with period $k$. Then all applications of rules from $\Rightarrow_k$ will result in the same word, i.e. $\{u : w \Rightarrow_k u\}$ is a singleton set.

As a consequence of this, the number of distinct descendants of $w$ with respect to $\Rightarrow$ is equal to the number of runs in $w$. 
Bounding from Above

Every reduction via $\Rightarrow$ removes at least one letter, thus there can be at most $n - 1$ steps in the reduction of a word of length $n$. So there are at most $\operatorname{duproots}(n) \leq \operatorname{runs}(n)^{n-3}$ different reductions.
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**Lemma**

If for two words $u, v \in \Sigma^*$ we have $\text{seq}(u) = \text{seq}(v)$, then also $\sqrt[♥]{u} = \sqrt[♥]{v} = \sqrt[♥]{\text{seq}(u)}$. 
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This means that we can first do all the possible reductions of the form $x^2 \rightarrow x$ for single letters $x$. For possible splits to different duplication roots we can assume that at least two letters are deleted in every step.
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Every reduction via \( \Rightarrow \) removes at least one letter, thus there can be at most \( n - 1 \) steps in the reduction of a word of length \( n \). So there are at most \( \text{duproots}(n) \leq \text{runs}(n)^{n-3} \) different reductions.

Lemma

If for two words \( u, v \in \Sigma^* \) we have \( \text{seq}(u) = \text{seq}(v) \), then also
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\sqrt[\heartsuit]{u} = \sqrt[\heartsuit]{v} = \sqrt[\heartsuit]{\text{seq}(u)}.
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This means that we can first do all the possible reductions of the form \( x^2 \to x \) for single letters \( x \). For possible splits to different duplication roots we can assume that at least two letters are deleted in every step.

This improves our upper bound to \( \text{runs}(n) \frac{n-3}{2} \).
10 versus 2 paths for the word $aabcbabcbcbc$, by first reducing one-letter squares from left to right. The direction of reductions is top to bottom.
• We construct an example of a sequence of words $w_n$, which are simply powers of a word $w$, namely $w_n := w^n$. The number of roots increases exponentially in $n$. 

Let $\rho$ be the morphism, which simply renames letters according to the scheme $a \rightarrow b \rightarrow c \rightarrow a$. Then $\rho(u)$ has the two roots $\rho(u_1)$ and $\rho(u_2)$; similarly, $\rho(\rho(u))$ has the two roots $\rho(\rho(u_1))$ and $\rho(\rho(u_2))$. 

$w = u \rho(u) \rho(\rho(u)) \rho(\rho(\rho(u))) = abcbabcbc \cdot d \cdot bcacbcaca \cdot d \cdot cabacabab \cdot d$. 

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$w = ud\rho(u)d\rho(\rho(u))d = abcbaabcba \cdot d \cdot bcacbcaca \cdot d \cdot cabacabab \cdot d$. 

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• \( w = ud\rho(u)d\rho(\rho(u))d = abcba bc bc \cdot d \cdot bcacbcaca \cdot d \cdot cabacabab \cdot d \).
Thus the duplication root of $w$ contains among others the three words

\[ w_a = abc \cdot d \cdot bca \cdot d \cdot cabacb \cdot d \]
\[ w_b = abc \cdot d \cdot bcacbcac \cdot d \cdot cab \cdot d \]
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We now need to recall that a morphism $h$ is called square-free, iff $h(v)$ is square-free for all square-free words $v$. Crochemore has shown that a uniform morphism $h$ is square-free iff it is square-free for all square-free words of length 3.
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The morphism we define now is \( \psi(x) := w_x \) for all \( x \in \{a, b, c\} \).
Checking Square-freeness

\[ \varphi(aba) = abcdbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
\[ \varphi(abc) = abcdbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
\[ \varphi(aca) = abcdbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
\[ \varphi(acb) = abcdbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
\[ \varphi(bab) = abcdbcacbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
\[ \varphi(bac) = abcdbcacbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
\[ \varphi(bca) = abcdbcacbcadcadcabacabdabcdbcacbacdabcdbcadcabacabd \]
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\[ \varphi(cab) = abcbabcdbcadcababdabcdbcadcabacabdbabcabcabcdbcadcab \]
\[ \varphi(cba) = abcbabcdbcadcababdabcdbcadcabacabdbabcabcabcdbcadcab \]
\[ \varphi(cbc) = abcbabcdbcadcababdabcdbcacbcadcadcabdbabcabcabcdbcadcabd, \]
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Then all the words in $\varphi(\text{pref}(t))$ are square-free, too. From the construction of $\varphi$ we know that for any word $z$ of length $i$ we can reach $\varphi(z)$ from $w^i$ by undoing duplications.
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Therefore $\varphi(\text{pref}(t)) \subseteq \sqrt[n]{w^+}$. For two distinct square-free words $t_1$ and $t_2$, also $\varphi(t_1) \neq \varphi(t_2)$. 

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Therefore $\varphi(\text{pref}(t)) \subseteq \sqrt{w^+}$. For two distinct square-free words $t_1$ and $t_2$, also $\varphi(t_1) \neq \varphi(t_2)$.

Finally, notice that for all positive $i \leq n$ we have $w^n \succ^* w^i$. 
Counting the Roots

We conclude that $\text{bduproots}_{\leq 30} \leq s$, where $s(n)$ is the number of ternary square-free words of length up to $n$. 

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This function’s value is not known, however, it was first bounded to $6 \times 1.032^n \leq s(n) \leq 6 \times 1.379^n$ by Brandenburg. A better lower bound was found by Sun $s(n) \geq 110^n {n \over 42}$.
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$w$ itself is of length $3|u| + 3 = 30$. So we see that $\text{bduproots}_{\leq 30}(n) \geq \frac{1}{30} 110^{\frac{n}{42}}$. 
The Bounds

Theorem

\[ \frac{1}{30} 110 \frac{n}{42} \leq \text{duproots}(n) \leq 2^n \text{ for all } n > 0. \]

Theorem

\[ \frac{1}{30} 110 \frac{n}{42} \leq \text{bduproots}_{\leq 30}(n) \leq \max\{812 \frac{n-3}{2}, 2^n\} \text{ for all } n > 0. \]

For ternary alphabet, the upper bound \(6 \cdot 1.379^n\) on the number of ternary words by Brandenburg can replace \(2^n\) in both Propositions.
Open Problems

- In how far does duproots($n$) depend on the alphabet size?
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- An example for exponential growth with only three letters.
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• In how far does \text{duproots}(n) depend on the alphabet size?
• An example for exponential growth with only three letters.
• How complicated is it to compute \text{duproots}(n)?