Algorythmics of posets generated by words over partially commutative alphabets

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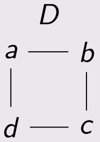
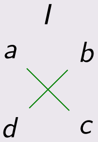
Concurrent alphabet

To the alphabet e.g. $\Sigma = \{ a, b, c, d\}$ we add the dependency relation or, equivalently, the indepedency relation





Words *abbaacd* and *abbcaad* are equivalent.

One word can be obtained from the other using transitions of the two consecutive letters.

Concurrent words and Poset

The dependency graph of a word

$$\Sigma = \{\, a, b, c, d \}\qquad D = \begin{array}{ccc} a & — & b \\ | & & | \\ d & — & c \end{array}$$

$$a \quad b \quad b \quad a \quad c \quad a \quad d$$

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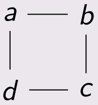

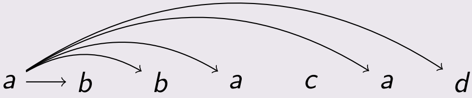

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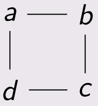
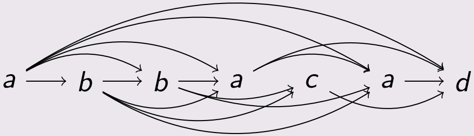
$D =$ 



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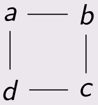
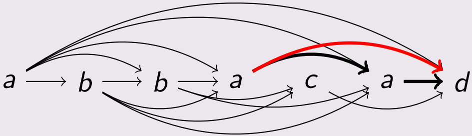
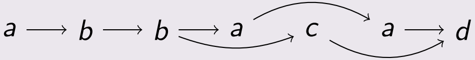




Transitive reduction (Hasse diagram)



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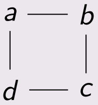

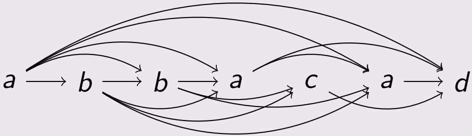

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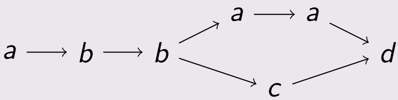
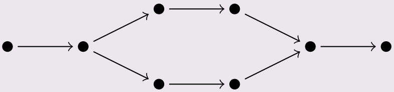


Concurrent words and Poset

Compressed version of a poset

Every partially ordered set is uniquelly identified by the Hasse diagram



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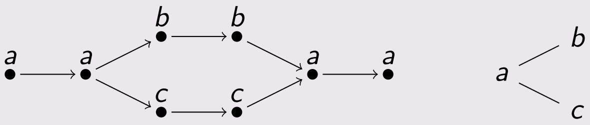

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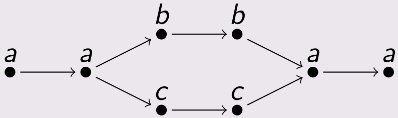

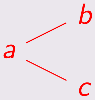

Every word obtained from topological sorting of this graph (the linearisation of a poset) may be used as a representative:

aabbccaa, *aabccbaa*, *aabcbcaa*, *aacbbcaa*, *aacbcbaa*, *aaccbbaa*

Definition of the problem

For given concurrent alphabet (Σ, D) and word $w \in \Sigma^*$ reproduce Hasse diagram of a poset (transitive reduction of word's dependency graph).

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- General algorithms of graph's transitive reduction have time complexity of $O(|V||E|) \sim O(n^3)$.
- It is possible to reduce this problem to the multiplication of boolean matrices - $O(n^{2.81\cdots})$.

Fact

If there exists an edge between $w_i = a$ and $w_j = b$ in the Hasse diagram, then letters a, b do not appear in the word w between indices i, j.

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Conclusions

- For every vertex there can be at most $k = |\Sigma|$ outgoing edges.
- For every vertex there can be at most k ingoing edges.
- Ingoing edges are determined by the last occurrences of the letters dependent with the label of a vertex.

Online algorithm of constructing Hasse diagram

Idea of the algorithm

$$D = \begin{array}{ccc} a & \text{---} & b \\ | & & | \\ d & \text{---} & c \end{array}, \quad w = adbcb$$

	D	V	Pos
a	(b, d, a)	$\{a\}$	1
b	(b, a, c)	$\{a, b\}$	3
c	(b, d, c)	$\emptyset$	0
d	(d, a, c)	$\{a, d\}$	2

$V = \emptyset$

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$$a_1 \longrightarrow d_2 \qquad b_3 \longrightarrow c_4 \quad (\text{arc } a_1 \to b_3)$$

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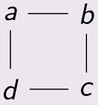
$D =$ 

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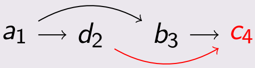

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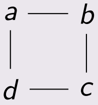
$D =$ 

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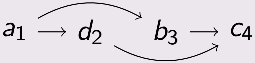

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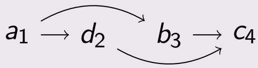

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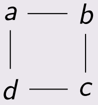
$D =$ 

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a	(b, d, a)	$\{a\}$	1
b	(c, b, a)	$\{a, b\}$	3
c	(c, b, d)	$\{a, b, c, d\}$	4
d	(c, d, a)	$\{a, d\}$	2

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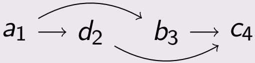

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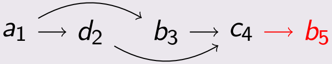

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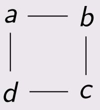
$D =$ 

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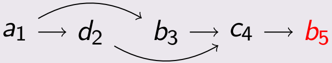

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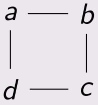
$D =$ 

$, \; w = adbcb$

	D	V	Pos
a	(b, d, a)	$\{a\}$	1
b	(b, c, a)	$\{a, b, c, d\}$	5
c	(b, c, d)	$\{a, c, d\}$	4
d	(c, d, a)	$\{a, d\}$	2

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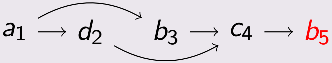


Algorithm 1: Hasse(w,(Σ, D))

```
foreach a ∈ Σ do
    L_a := 0; V_a := ∅;
for i := 1 to n do
    a := w_i; V := ∅;
    foreach b ∈ D_a in the order of last occ. do
        if L_b ≠ 0 and b ∉ V then
            Add the edge w_{L_b} → w_i;
            V := V ∪ V_b;
    foreach b ∈ Σ do
        V_b := V_b \ {a};
    foreach b ∈ D_a do
        Move a to the beginning of D_b;
    V_a := V ∪ a; L_a := i;
```

Online algorithm of constructing Hasse diagram

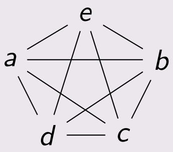
Time complexity

Given algorithm has time complexity $O(nk^2)$, where k is a size of the alphabet and n is the length of the word.

Definition of the problem

For given concurrent alphabet (Σ, D) and a natural number n generate every pairwise nonequivalent words of length n (lexicographically minimal representatives of posets).

Example



aa	*ba*	*ca*	*da*	*ea*
ab	*bb*	*cb*	*db*	*eb*
ac	*bc*	*cc*	*dc*	*ec*
ad	*bd*	*cd*	*dd*	*ed*
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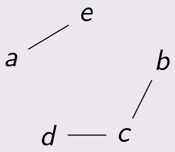

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ad	bd	cd	dd	ed
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Example

e

a b

d c

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ac	*bc*	*cc*	*dc*	*ec*
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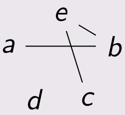

$ebceee \longrightarrow ebdaaa$

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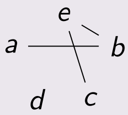

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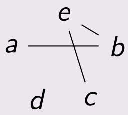

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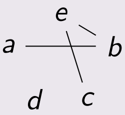

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"Naive" algorithm

Preprocessing: $D_{min} =$

a	b	c	d	e
a	a	c	d	b

1. Find the latest letter w_i that is different that the largest
2. Change w_i to the next letter
3. Find out if actual prefix is minimal
4. Using the table D_{min} generate the sufix

Algorithm "with oracle"

$$V_1(a) = a$$

$$V_i(a) = \begin{cases} a & \text{if } aDw_{i-1} \\ \max(w_{i-1}, V_{i-1}(a)) & \text{otherwise} \end{cases}$$

- Cost of the preprocessing - $O(kn)$
- Finding out if the change is valid in time $O(1)$
- While generate sufix we must actualise the oracle

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- Finding out if the change is valid in time $O(1)$
- While generate sufix we must actualise the oracle
- We have at most k different colums of an oracle, the cost of actualisation is $O(k \min(k, \#suff) + \#suff)$

Complexity comparison

Work of both algorithms is dependent on the length of changing suffix. Pesymistic cost of naive algorithm

$$O(k\ \#pref + \#suff)$$

cost of algorithm "with oracle"

$$O(k \min(k, \#suff) + \#suff)$$

Complexity comparison

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$$O(n)$$

cost of algorithm "with oracle"

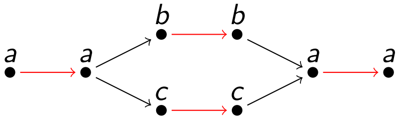
$$O(\#suff)$$

Definition of the problem

For given Hasse diagram write out in lexicographical order all linearisations.

- There are efective algorithms that do not preserve lexicographic order.
- Using technic similar to the algorithm of Hasse diagram generation, we can efectively write out linearisations in lexicographical order.

Combining neighbour actions into blocks



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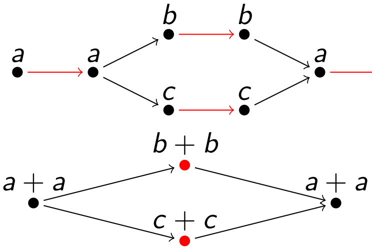

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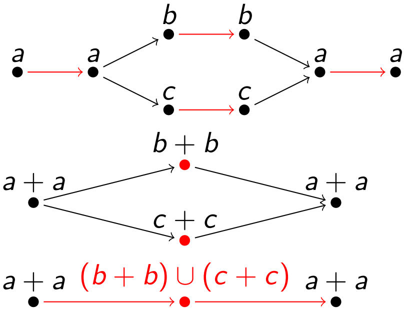

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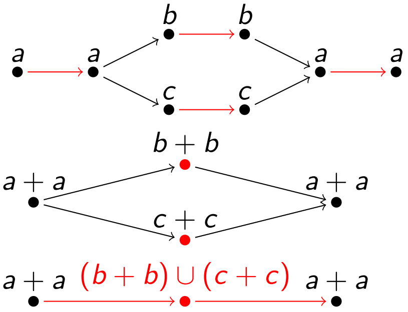


$$a + a + ((b + b) \cup (c + c)) + a + a$$

Series-parallel graphs

Definition

Series-parallel graph is a graph consisting of single vertex and no edges or is constructed form two series-parallel graphs using series and parallel composition.

Fact

Diagrams that may be compressed in presented way are precisely the series-parallel graphs.

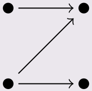
Fact

Many algorithmic problems are much simpler in the case of series-parallel graphs (posets).

N-free graphs

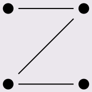
Fact

Graph is series-parallel if and only if it is N-free



Fakt

Hasse diagram of poset, which alphabet is N-free (P4-avoiding) is also N-free.



Thank you
for your attention.