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Faster Longest Common Extension on Compressed Strings and Applications

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$p = 6$ $q = 34$

`I argue string algorithms at Prague stringology`

$$\textbf{LCE}(6, 34) = 9$$

Background & Motivation

- ✓ LCE has numerous applications, e.g., approximate pattern matching, computing palindromes, computing approximate repeats.
- ✓ A string T of length u can be preprocessed in $O(u)$ time and space so that each LCE query can be answered in $O(1)$ time [Demaine et al.].
- ✓ However, the $O(u)$ complexity can be prohibitive for large-scaled text.
- ✓ To save preprocessing time and space, we consider **LCE on grammar-compressed text**.

Straight Line Program (SLP)

Definition

An SLP is a sequence of n productions

$$X_1 \to expr_1,\ X_2 \to expr_2, \cdots,\ X_n \to expr_n$$

- $expr_i = a \qquad (a \in \Sigma)$
- $expr_i = X_l X_r \quad (l, r < i)$

✓ An SLP is a CFG in the Chomsky normal form which derives a single string.

✓ SLPs model outputs of grammar-based compression algorithms (e.g., Re-pair, LZ78, LZDF, OLCA, etc).

Straight Line Program (SLP)

n : size (# of productions) of a given SLP S

h : height of the derivation tree of S

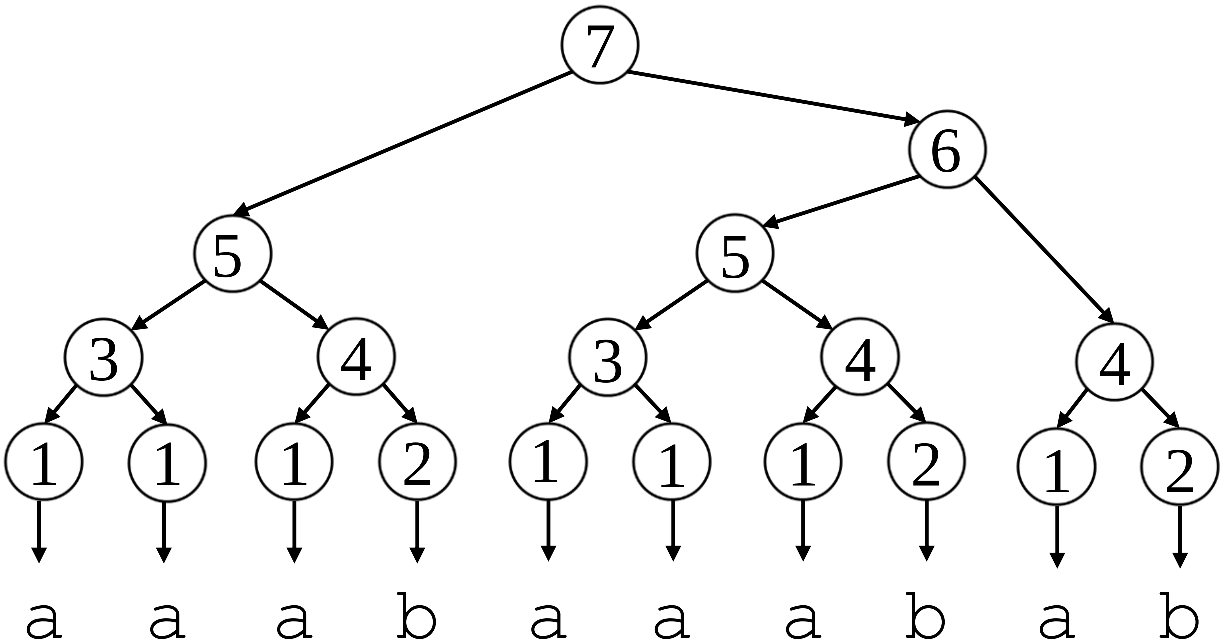
u : length of the uncompressed string T represented by SLP S

Example of SLP

SLP S

$X_1 \rightarrow$ a
$X_2 \rightarrow$ b
$X_3 \rightarrow X_1 X_1$
$X_4 \rightarrow X_1 X_2$
$X_5 \rightarrow X_3 X_4$
$X_6 \rightarrow X_5 X_4$
$X_7 \rightarrow X_5 X_6$

Derivation tree of SLP S



Example of SLP

SLP S

$$\left.\begin{array}{l}
X_1 \rightarrow \mathtt{a} \\
X_2 \rightarrow \mathtt{b} \\
X_3 \rightarrow X_1\, X_1 \\
X_4 \rightarrow X_1\, X_2 \\
X_5 \rightarrow X_3\, X_4 \\
X_6 \rightarrow X_5\, X_4 \\
X_7 \rightarrow X_5\, X_6
\end{array}\right\} \boxed{n}$$

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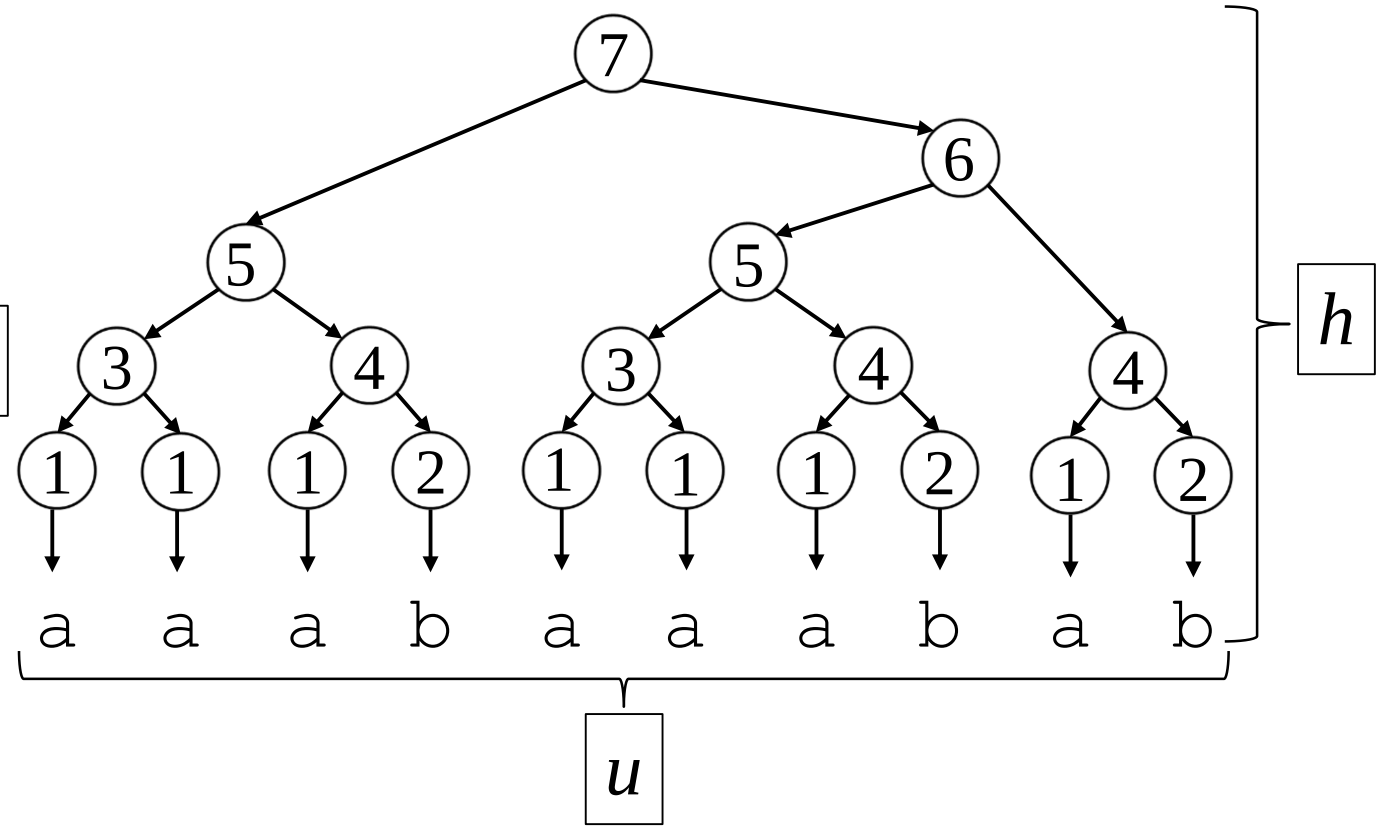

Example of SLP

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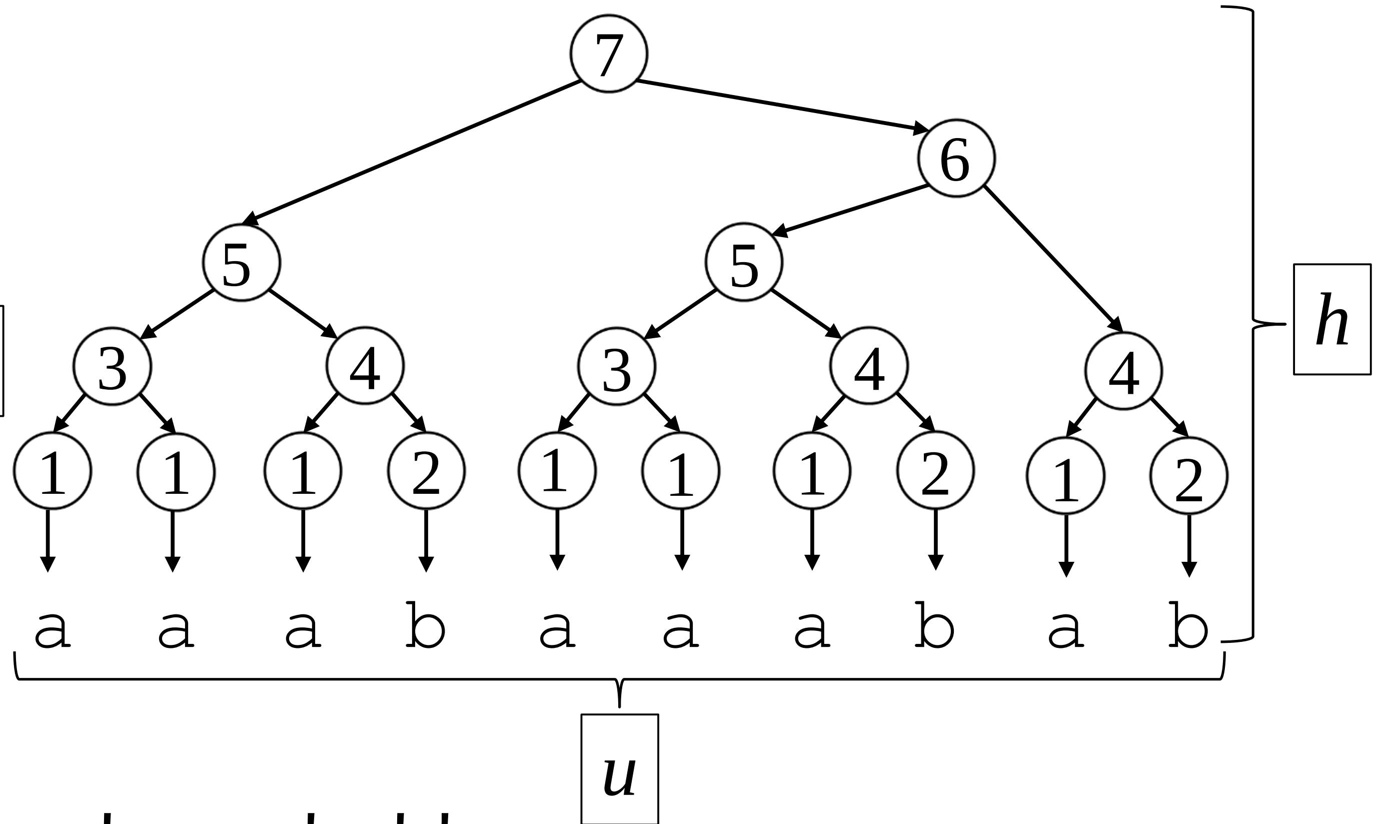
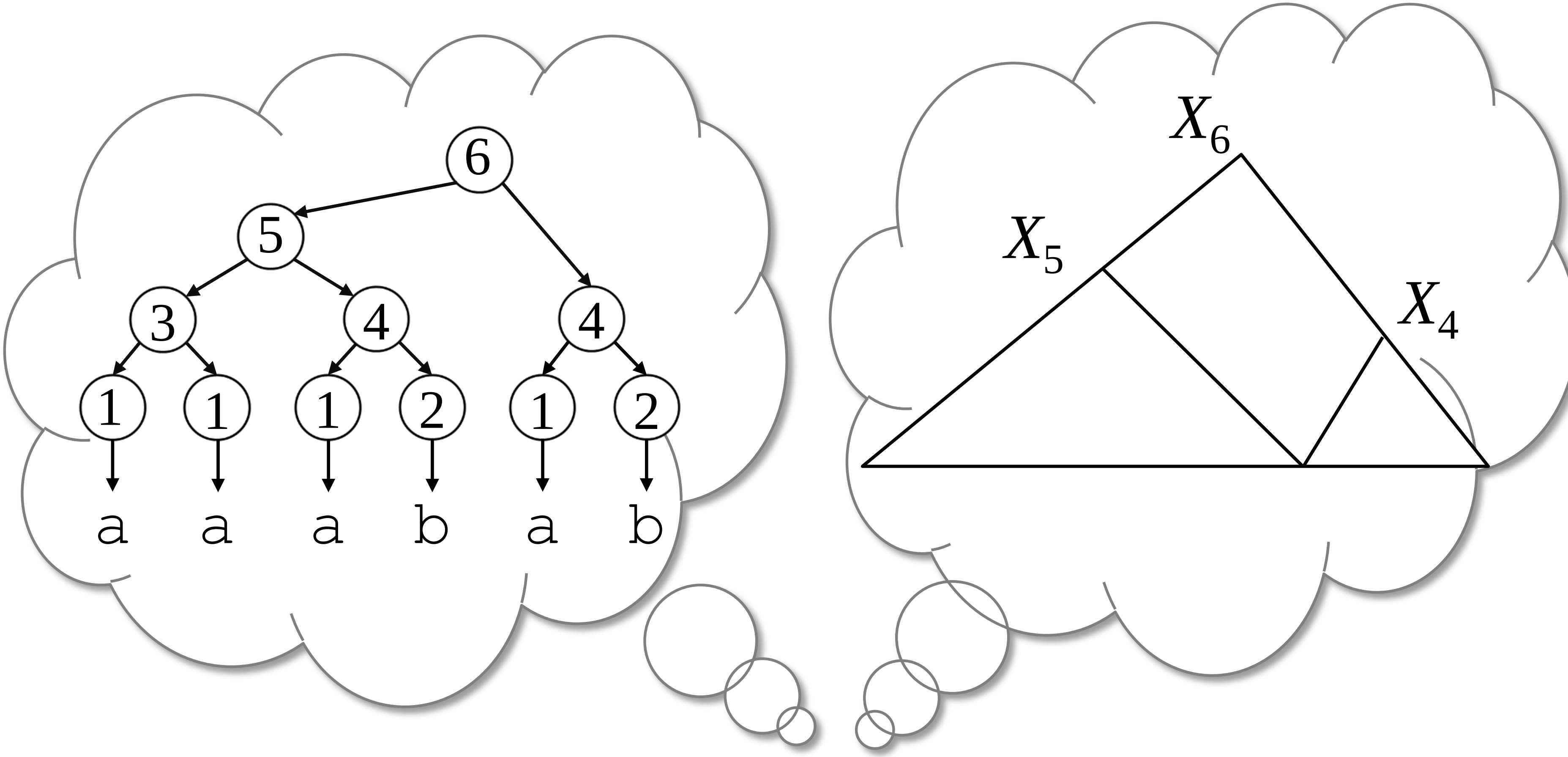


- ✓ $\log_2 u \leq h \leq n$ always holds.
- ✓ u can be exponential in n (e.g. consider string a^u).
  - ➢ Hence, $O(\mathrm{poly}(n))$ solutions are of significance.

Important Remarks

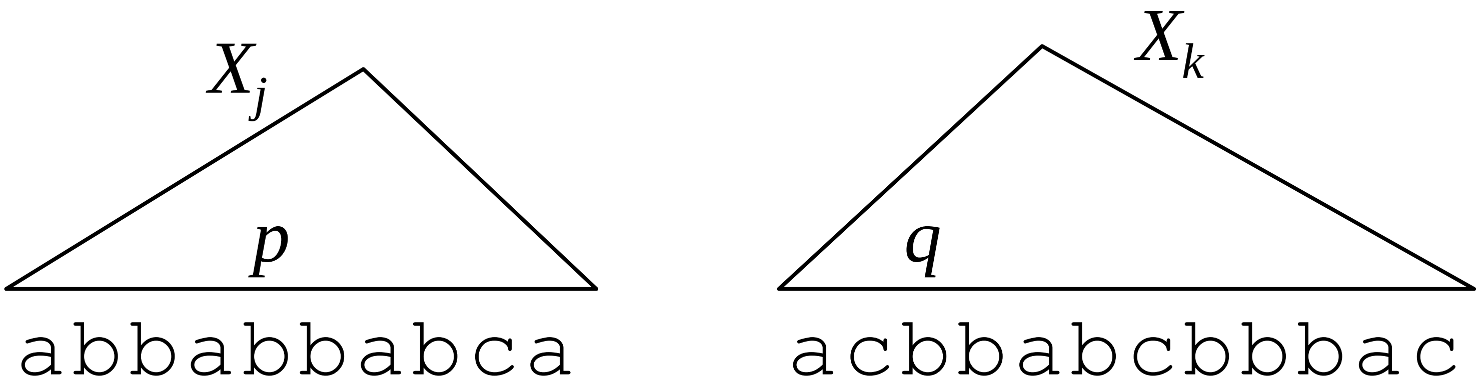


- ✓ Derivation trees are only **imaginary** (used only for explanations) and are <u>never constructed explicitly</u>.

Longest Common Extension on SLP

Problem 1 (grammar compressed LCE)

Preprocess an input SLP $S = \{X_i \rightarrow expr_i\}_{i=1}^{n}$ so that subsequent longest common extension queries $\mathbf{LCE}(X_j, X_k, p, q)$ can be answered quickly.



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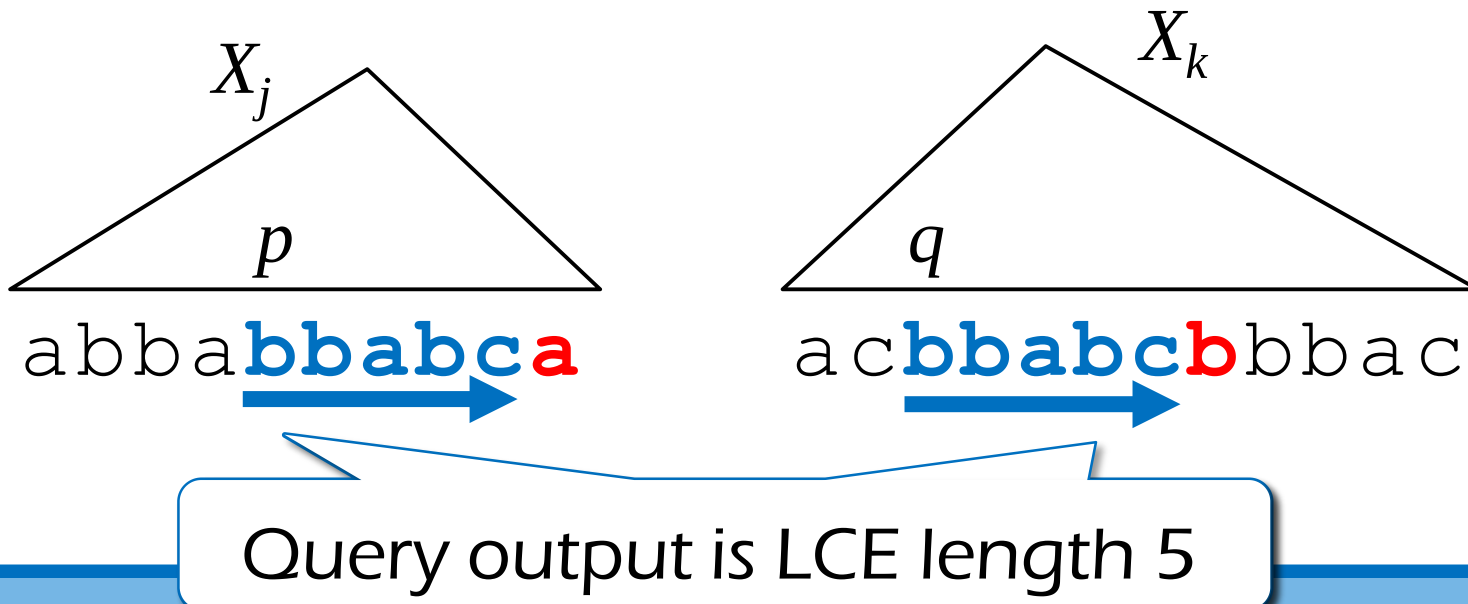

What is the difficulty?

- ✓ We are not allowed to expand the SLP (compressed text), since this takes $O(2^n)$ time in the worst case.
- ✓ But we want to know the length of the longest common extension!

LCE algorithms on SLPs

Algorithms	Query time	Preprocessing time	Space
Folklore	$O(hL)$	$O(n)$	$O(n)$
(extended) Miyazaki et al. '97	$O(hn^2)$	$O(n^4)$	$O(n^2)$
(extended) Lifshits '07	$O(hn^2)$	$O(hn^2)$	$O(n^2)$
I et al. '15	$O(h \log u)$	$O(hn^2)$	$O(n^2)$
Bille et al. '15 (randomized)	$O(\log u + \log^2 L)$	N/A	$O(n)$

n: size of SLP
u: length of uncompressed string T
h: height of SLP derivation tree
L: LCE length (output)
z: size of LZ77 factorization of T

- $\log u \leq h \leq n$
- $L = O(u)$
- $\log^* u = o(\log u)$
- $z \leq n$ (due to Rytter '03)

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This work	$O(\log u + \log^* u \log L)$	$O(n \log\log n \log^* u \log u)$	$O(n + z \log^* u \log u)$

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Logstar (iterated logarithm)

Definition

The **logstar** of a positive integer u, denoted $\log^* u$, is the number of times the logarithm function needs to be iteratively applied to u until the result becomes less than or equal to 1.

- ✓ The logstar is a <u>very slowly growing function</u>, e.g., $\log^* 2^{65536} = 5$.

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n: size of SLP

u: ...essed strin...

z: si... ...ization of T

- $\log u \le h \le$ ...
- $z \le n$ (due t...

Fastest deterministic queries

Fastest preprocessing

Smallest in many cases

Our strategy

- ✓ All previous algorithms work on the SLP derivation trees of two query non-terminals.

- ✓ Our new algorithm does NOT work on the SLP derivation trees.
- ✓ Instead, we construct a different tree of logarithmic height, based on
  - ➢ locally consistent parsing
  - ➢ signature encoding.

Locally consistent parsing

Lemma 1 [Mehlhorn et al., Alstrup et al.]

For any integer string $Y \in \{1..m\}^*$ in which no adjacent elements are equal (i.e. $Y[i] \neq Y[i{+}1]$), there is a bit string d of length $|Y|$ such that

1. no 1's appear consecutively;
2. at most three 0's appear consecutively;
3. each $d[i]$ is determined locally, i.e., by $Y[i-\Delta_L \ldots i-1]$ and $Y[i \ldots i+\Delta_R]$, where $\Delta_L \leq \log^* m + 6$ and $\Delta_R \leq 4$;
4. d can be computed in $O(|Y|)$ time.

Locally consistent parsing

$$Y = 1,2,3,5,2,3,4,2,5,1,2,3,5,2,3,4,2,5$$
$$d = 1,0,1,0,0,1,0,0,0,1,0,1,0,1,0,1,0,0$$

Locally consistent parsing

$$\Delta_L \qquad \Delta_R$$

$$Y = 1,2,3,5,2,3,4,2,5,1,2,3,5,2,3,4,2,5$$

$$d = 1,0,1,0,0,1,0,0,\mathbf{0},1,0,1,0,1,0,1,0,0$$

$$\Delta_L \le \log^* m + 6$$

$$\Delta_R \le 4$$

Locally consistent parsing

$$Y = \boxed{1,2}, \boxed{3,5,2}, \boxed{3,4,2,5}, \boxed{1,2}, \boxed{3,5}, \boxed{2,3}, \boxed{4,2,5}$$

$$d = 1,0,1,0,0,1,0,0,0,1,0,1,0,1,0,1,0,0$$

✓ Using the bit string d, any integer string Y can be uniquely decomposed in linear time into blocks of length 2-4.

Signature encoding [Mehlhorn et al. '97]

- ✓ Iteratively apply locally consistent parsing to input string T until a single integer is obtained.

$T =$ a b c a c a b b c a b a c c c a

Signature encoding [Mehlhorn et al. '97]

✓ Iteratively apply locally consistent parsing to input string T until a single integer is obtained.

Each character is assigned to a unique integer called a signature.

	1	2	3	1	3	1	2	2	3	1	2	1	3	3	3	1
$T =$	a	b	c	a	c	a	b	b	c	a	b	a	c	c	c	a

Signature encoding [Mehlhorn et al. '97]

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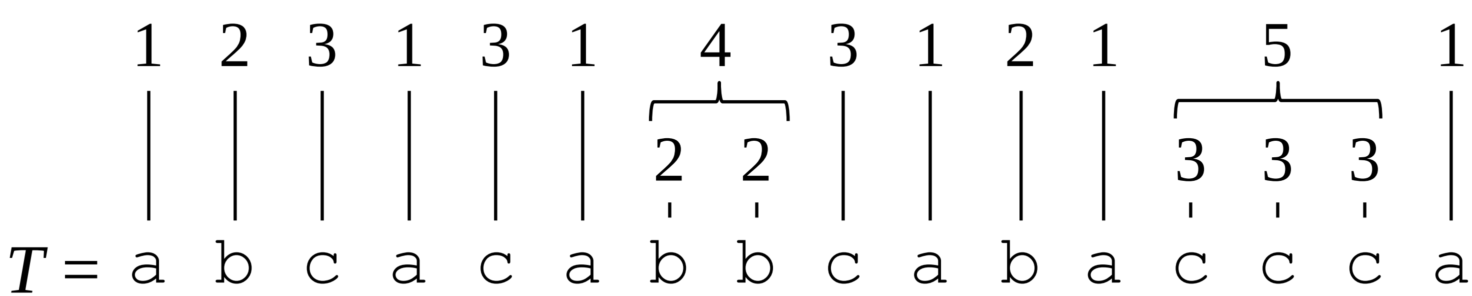
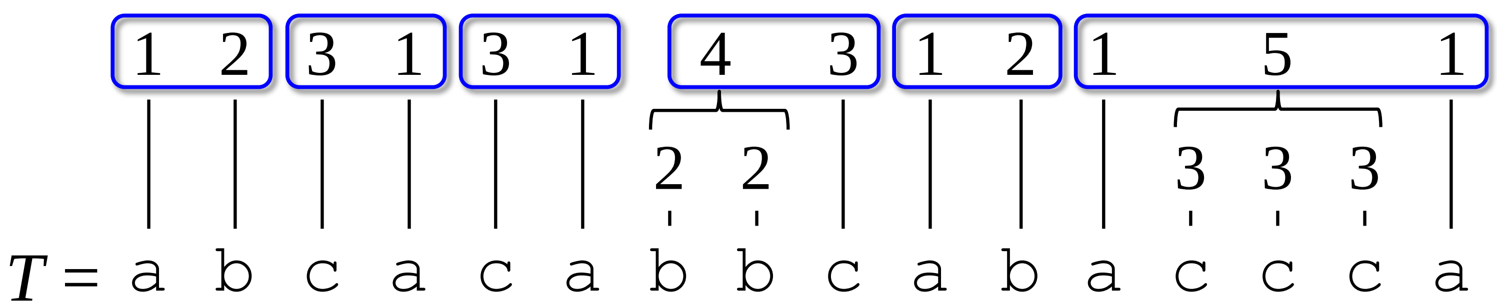
Maximal run of the same signatures is assigned to a new signature.



Signature encoding [Mehlhorn et al. '97]

✓ Iteratively apply locally consistent parsing to input string T until a single integer is obtained.

Apply locally consistent parsing to this string.



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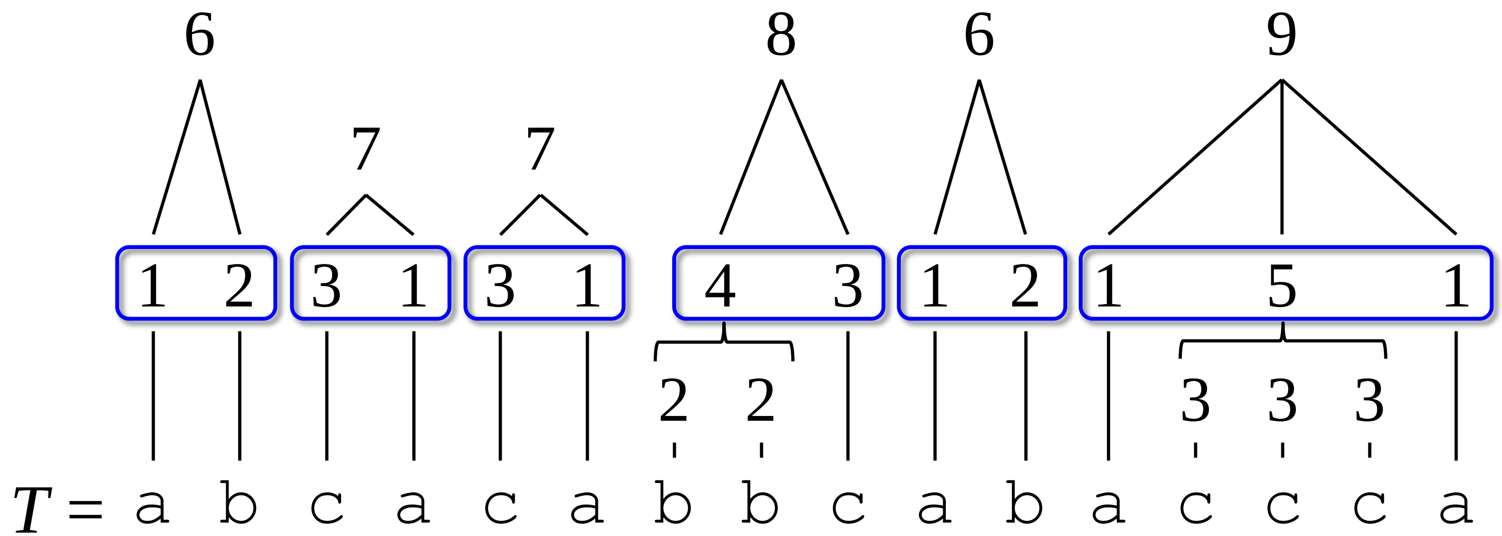

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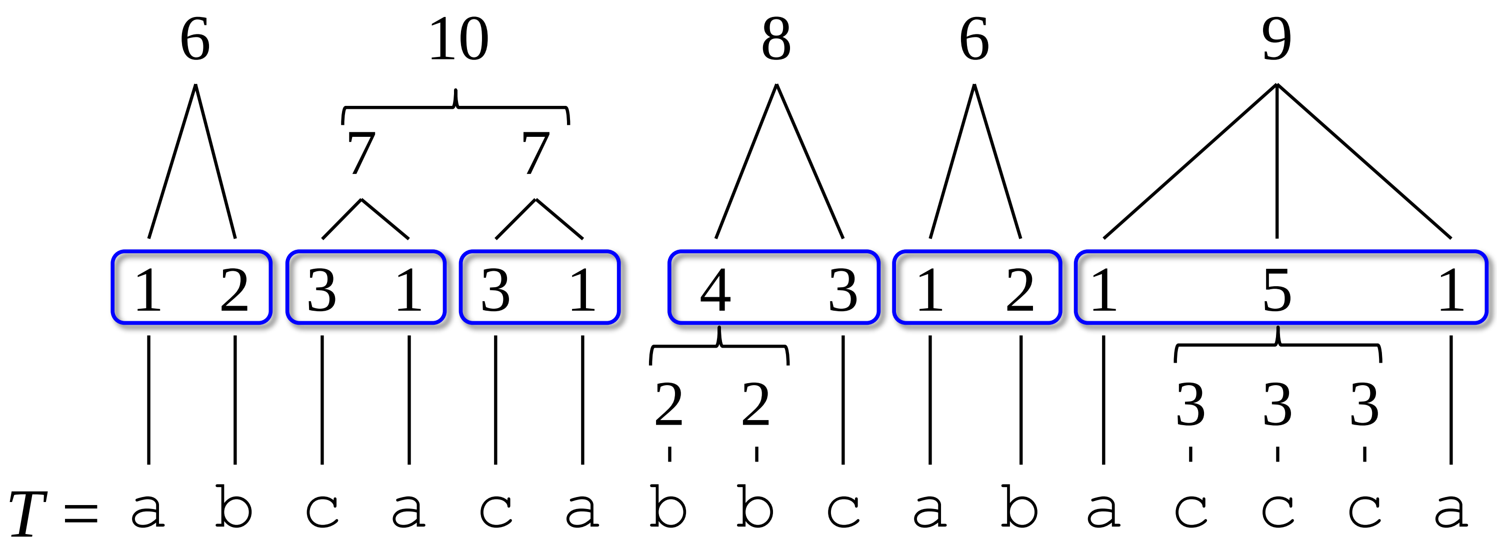

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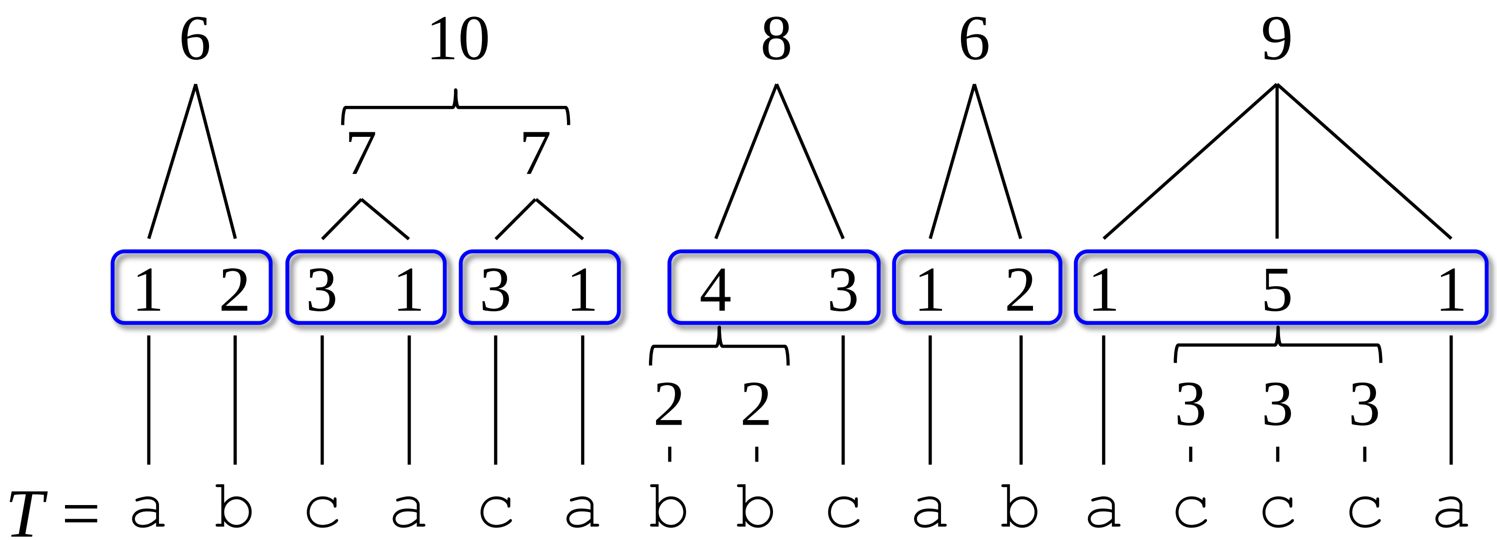

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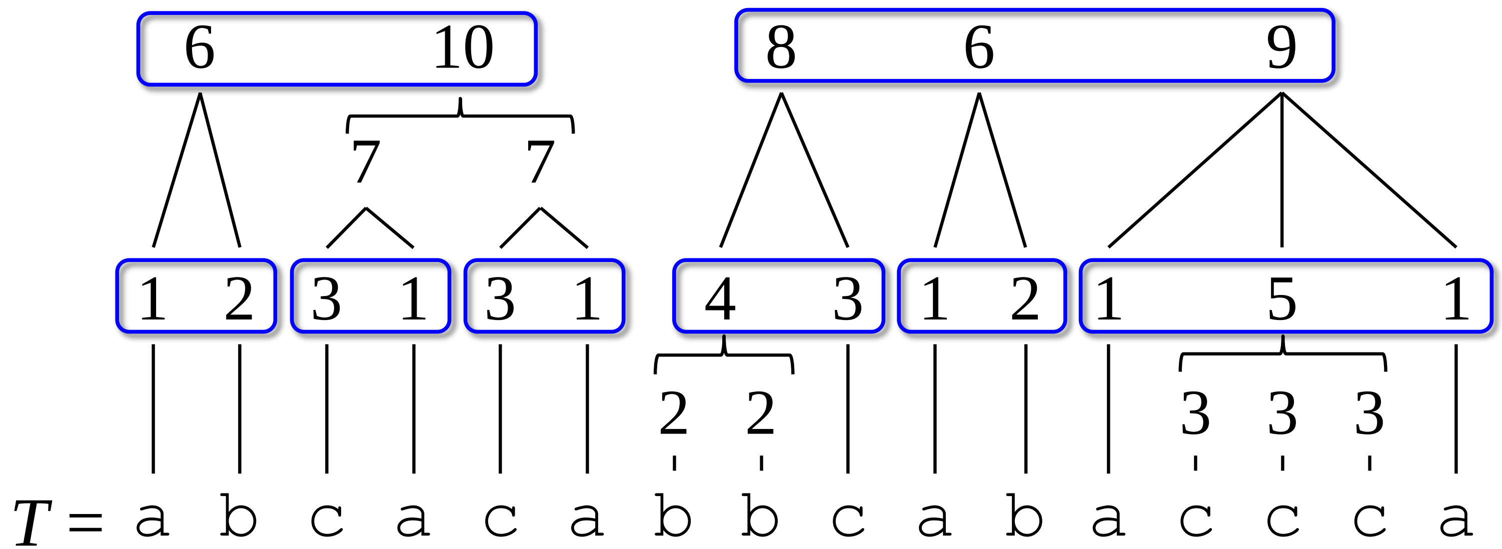

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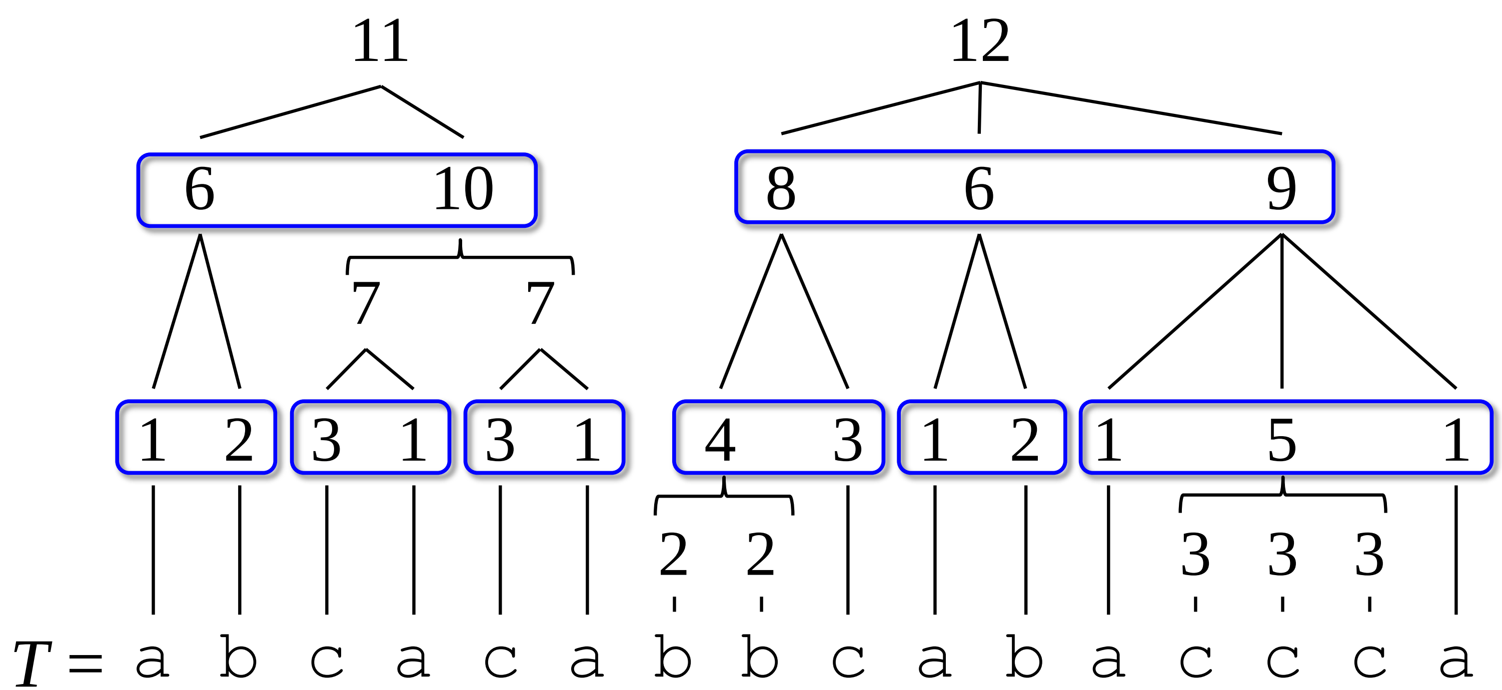

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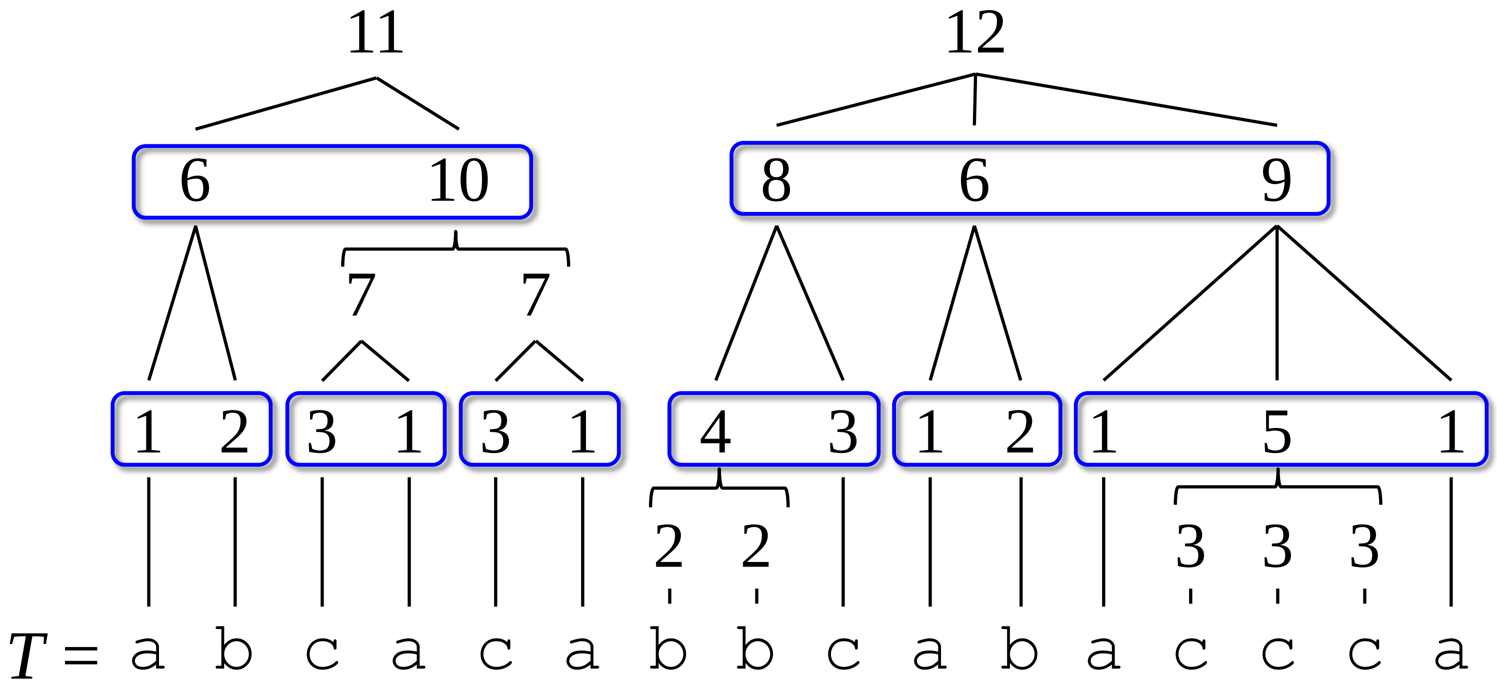

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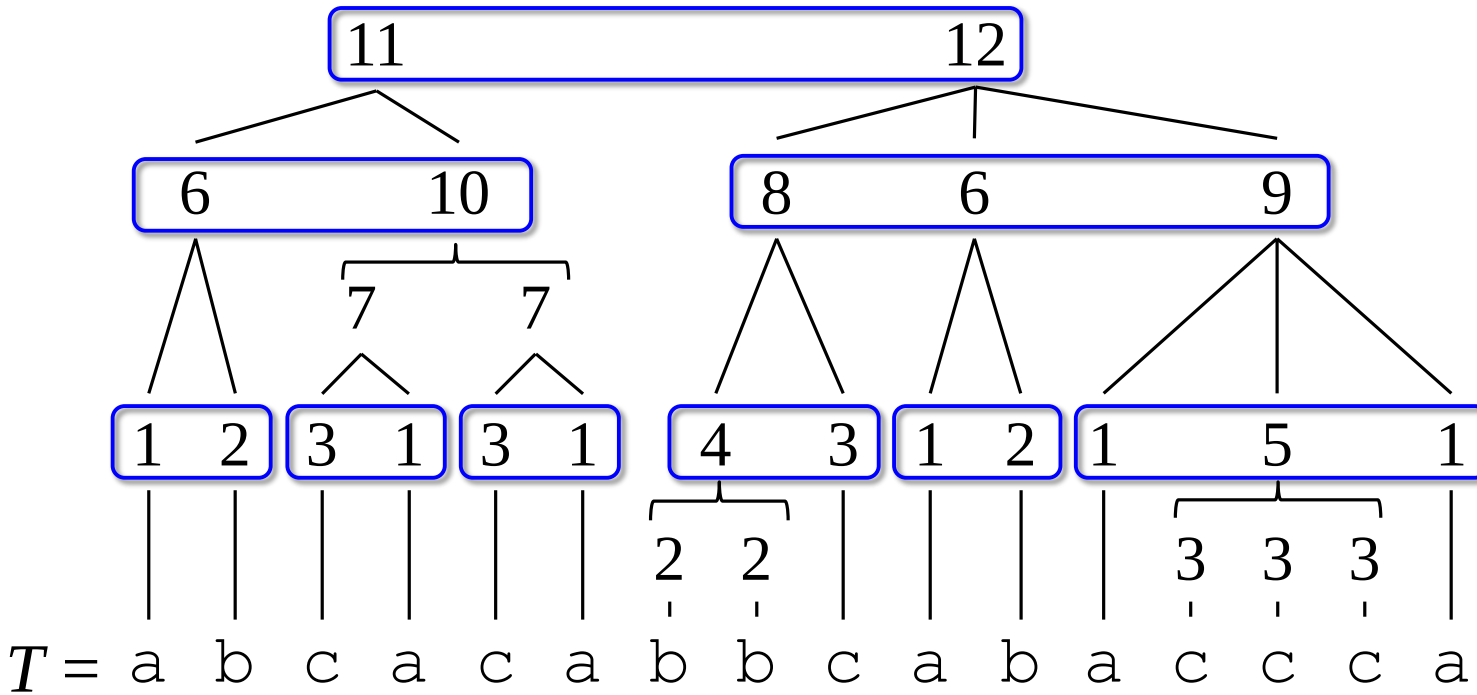

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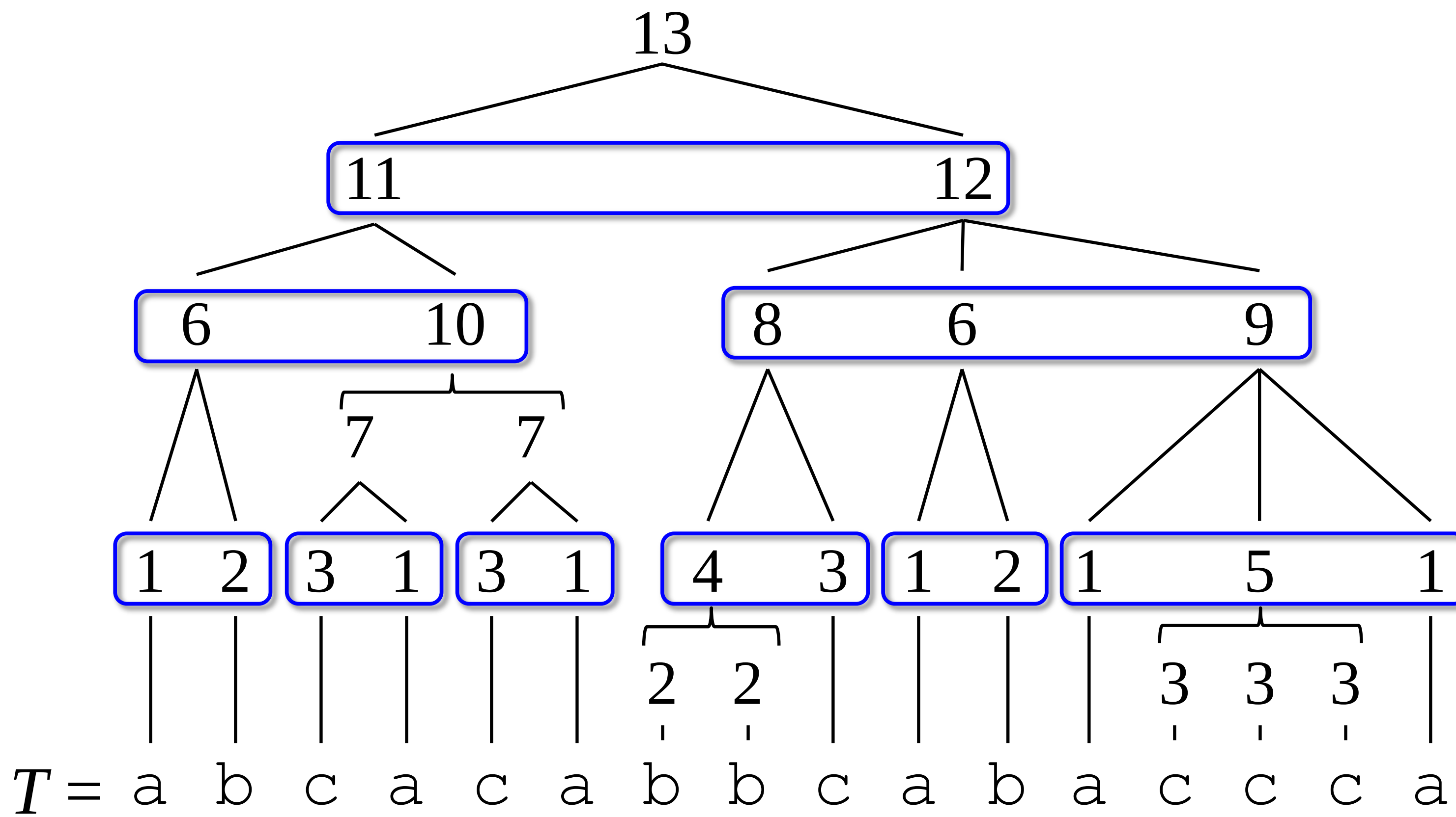
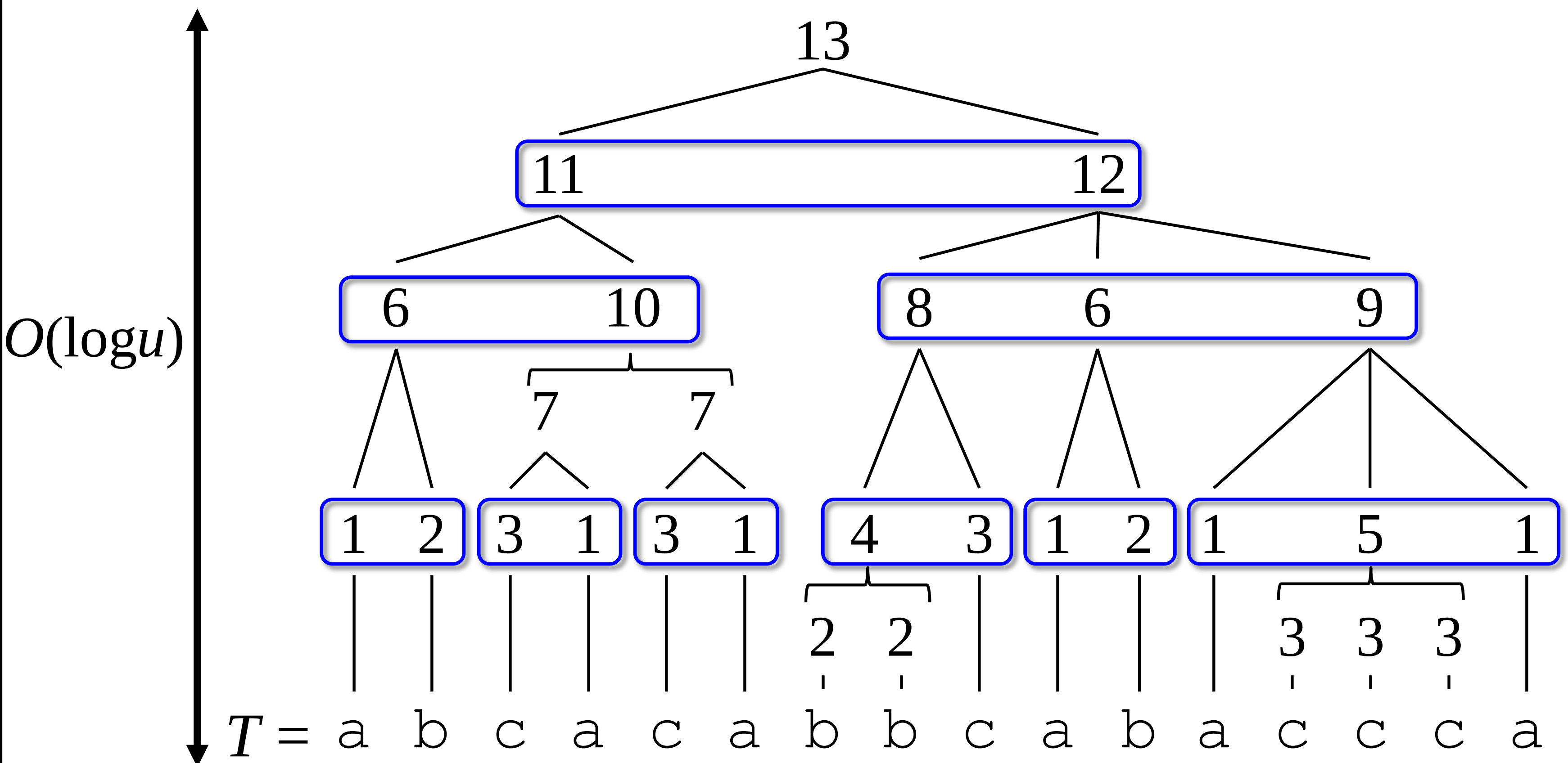


Signature encoding [Mehlhorn et al. '97]

- ✓ The height of this tree, called the signature tree, is $O(\log u)$, where $u = |T|$.



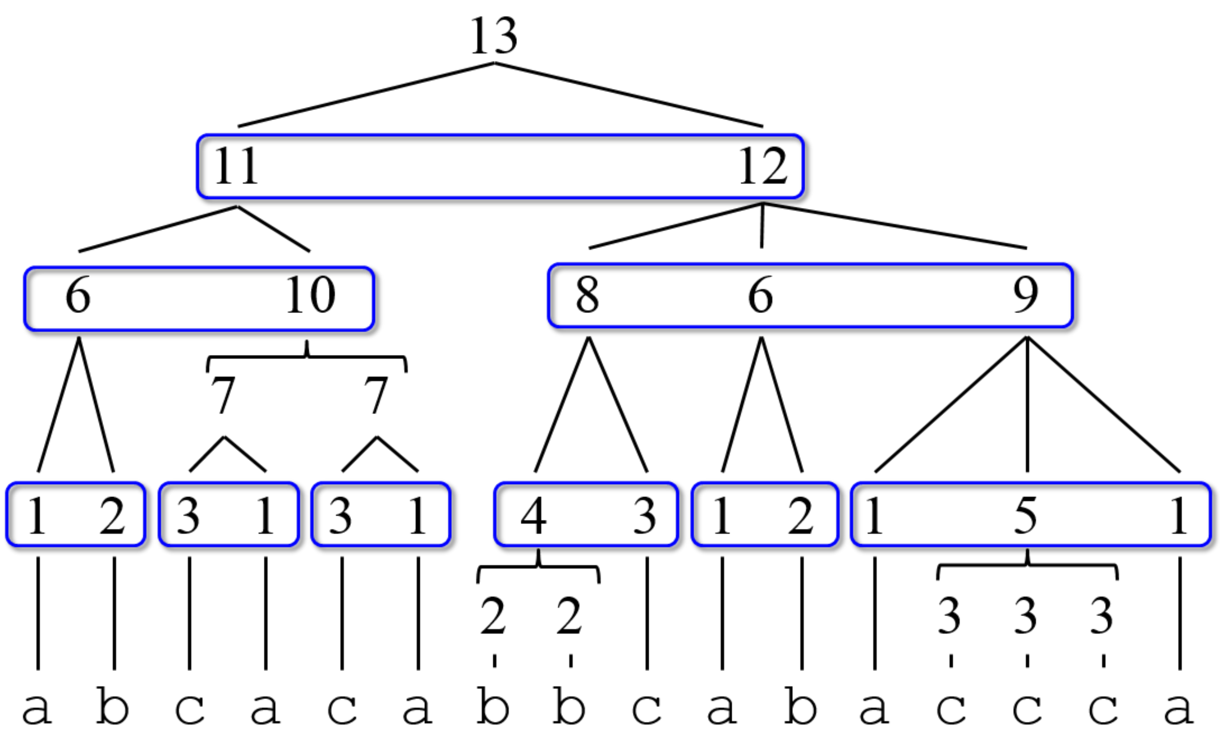
Signature encoding [Mehlhorn et al. '97]

✓ The dictionary D_T of signatures is the **signature encoding** of input string T.

D_T

$13 \rightarrow 11, 12$
$12 \rightarrow 8, 6, 9$
$11 \rightarrow 6, 10$
$10 \rightarrow 7^2$
$9 \rightarrow 1, 5, 1$
$8 \rightarrow 4, 3$
$7 \rightarrow 3, 1$
$6 \rightarrow 1, 2$
$5 \rightarrow 3^3$
$4 \rightarrow 2^2$
$3 \rightarrow$ c
$2 \rightarrow$ b
$1 \rightarrow$ a

signature tree of T



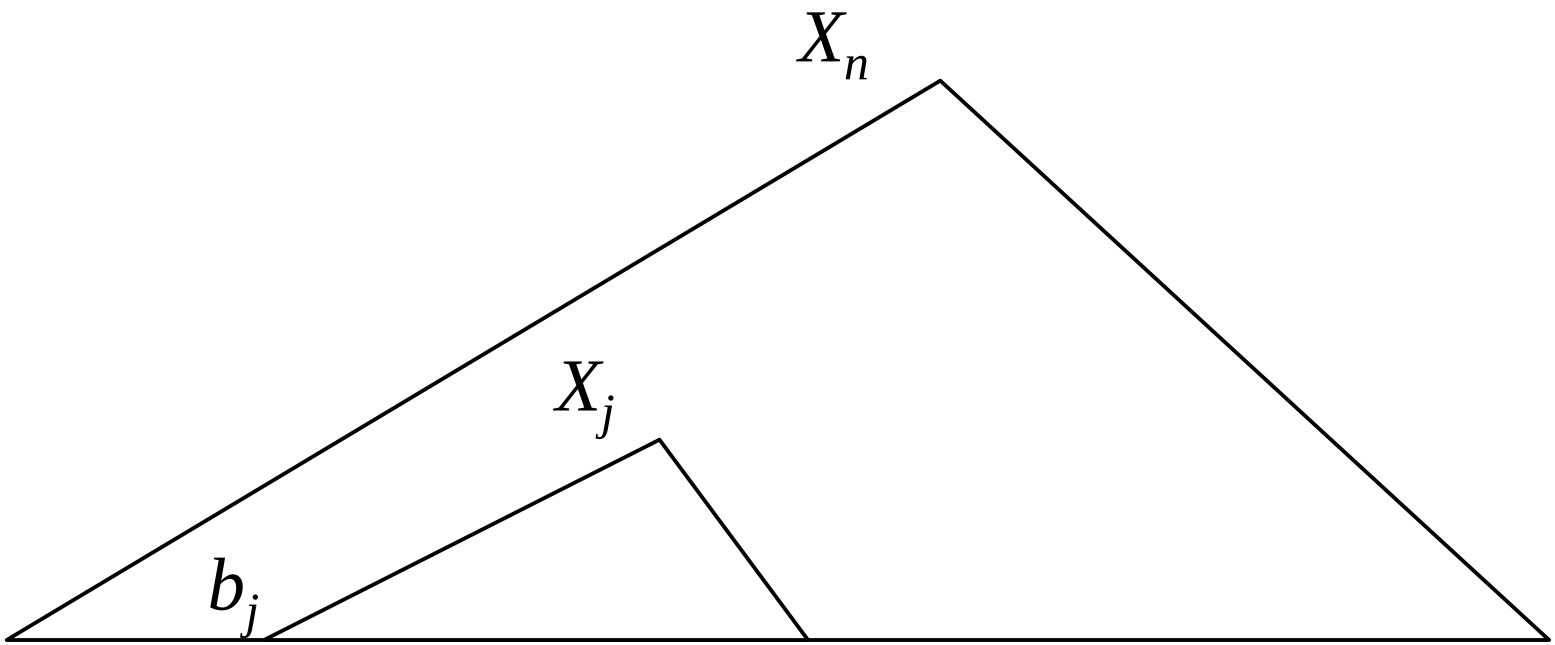
Faster LCE algorithm on SLP

Lemma 2 (Faster LCE on SLP)

Given the signature encoding D_T of string T of length u, we can compute $\mathbf{LCE}(X_j, X_k, p, q)$ for any variables X_j, X_k and positions p, q in $O(\log u + \log^* u \log L)$ time, where L is the answer to the query (LCE length).

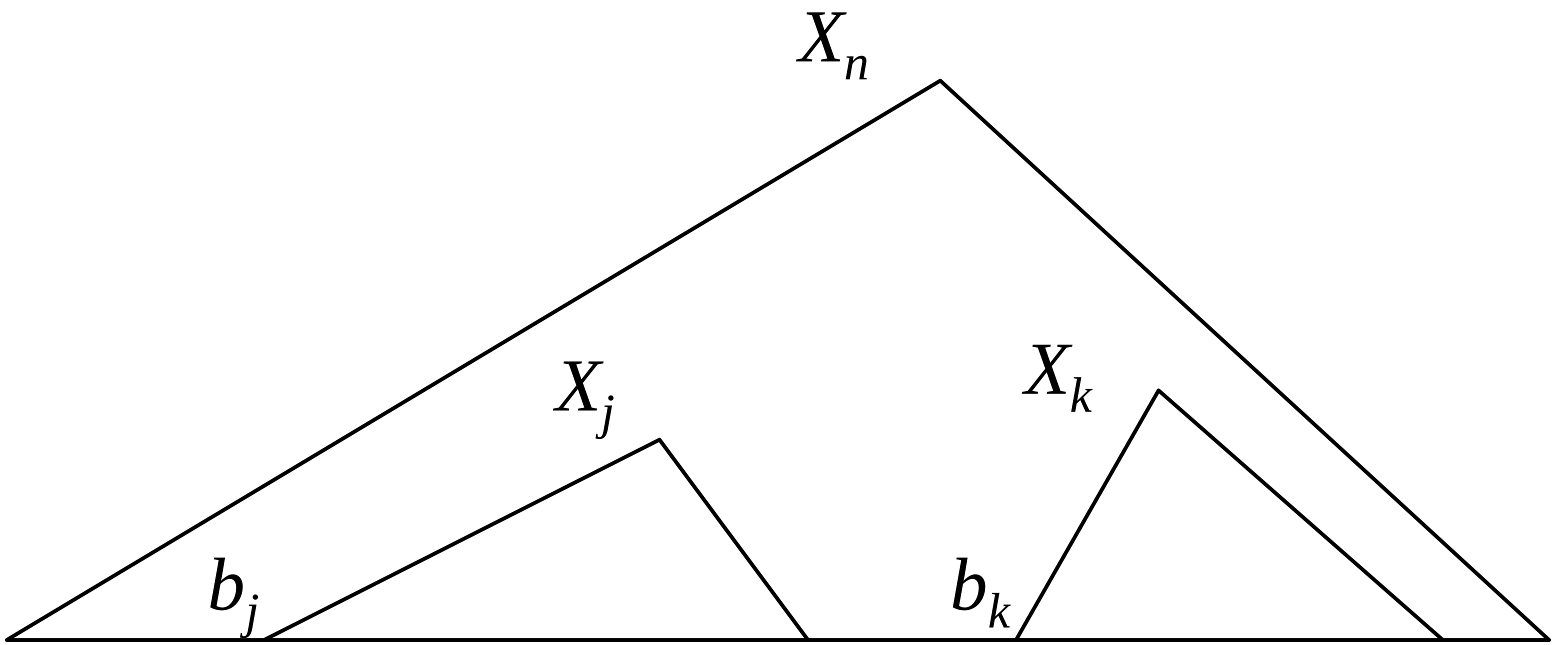
Faster LCE algorithm on SLP

1. For every non-terminal X_j, we precompute and store its occurrence b_j in the derivation tree of X_n.



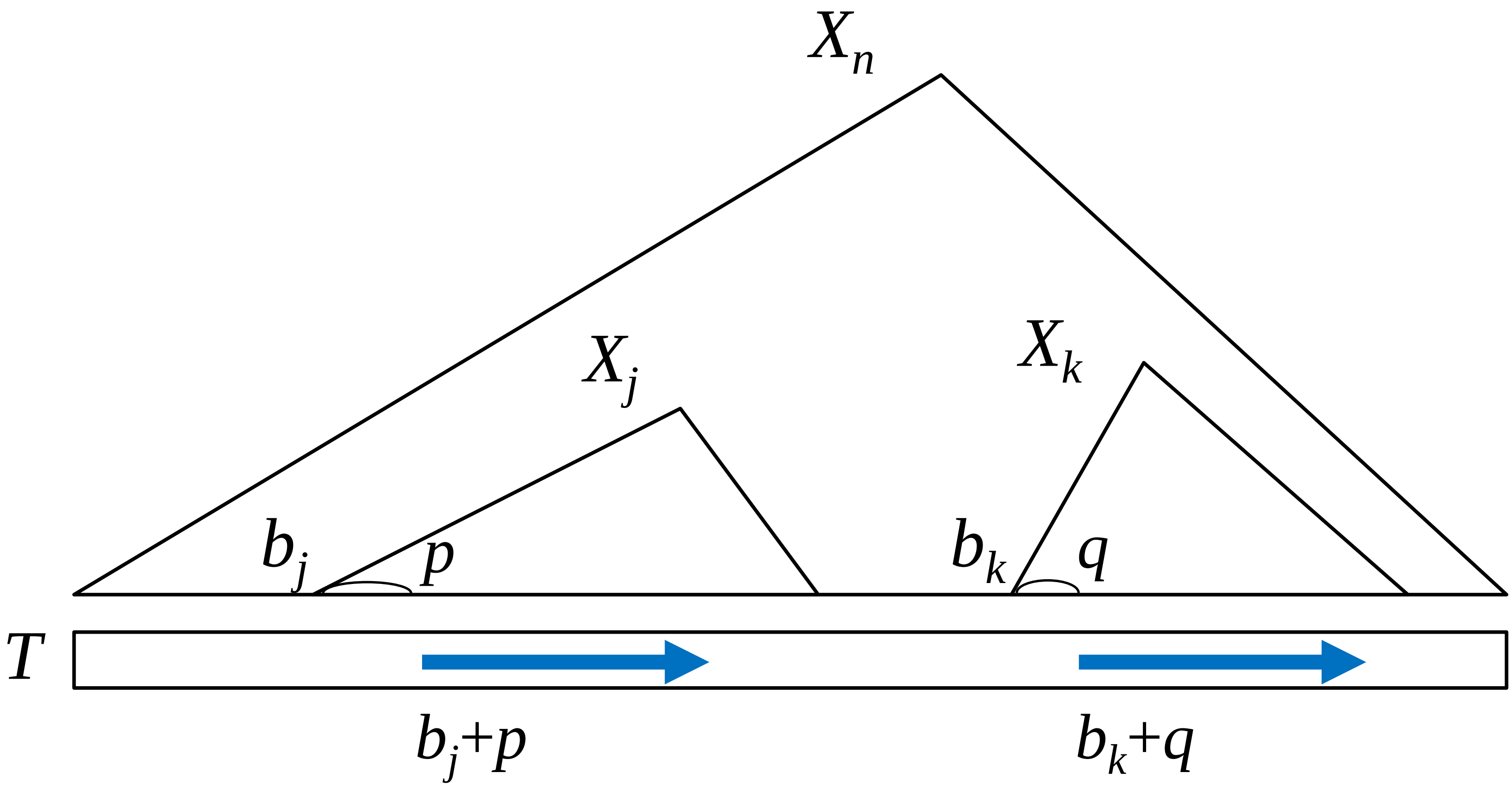
Faster LCE algorithm on SLP

2. Given query variables X_j and X_k for LCE, we retrieve b_j and b_k.



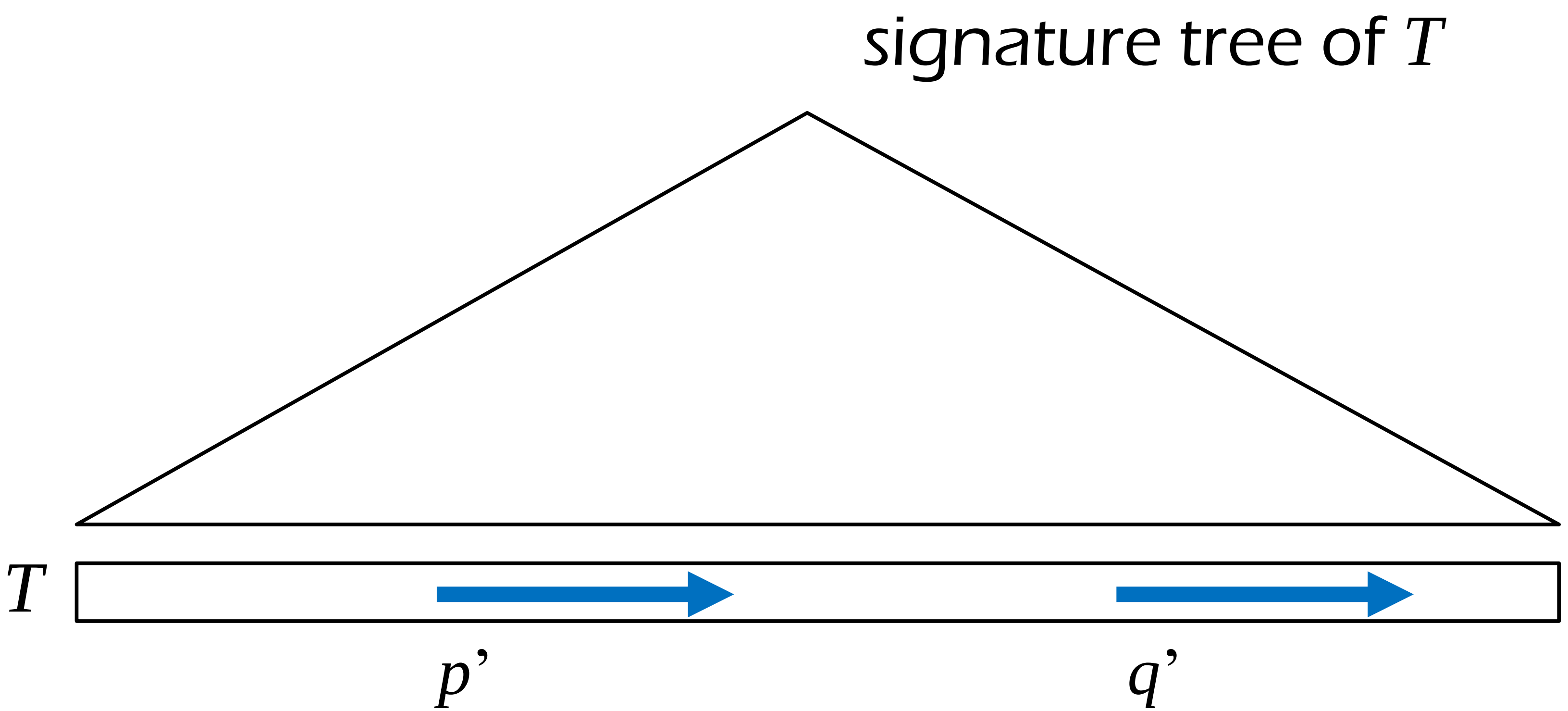
Faster LCE algorithm on SLP

3. Since the last variable X_n derives string T, $\mathbf{LCE}(X_j, X_k, p, q)$ reduces to $\mathbf{LCE}(b_j+p, b_k+q)$ on string T.



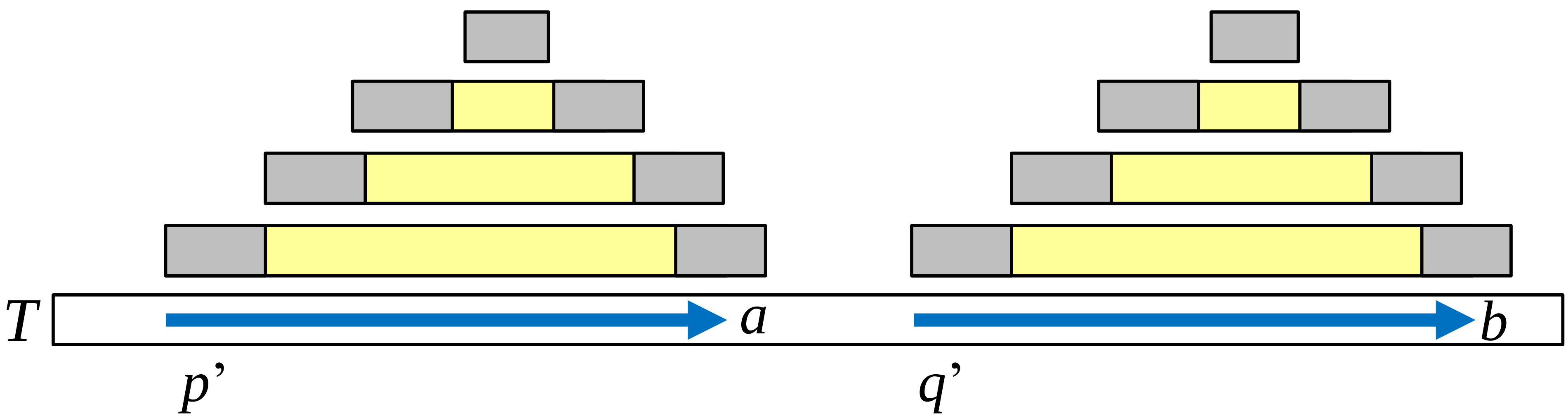
Faster LCE algorithm on SLP

4. We turn attention to the <u>signature tree</u> of T, and compute $\mathbf{LCE}(p', q')$ there, where $p' = b_j + p$ and $q' = b_k + q$.



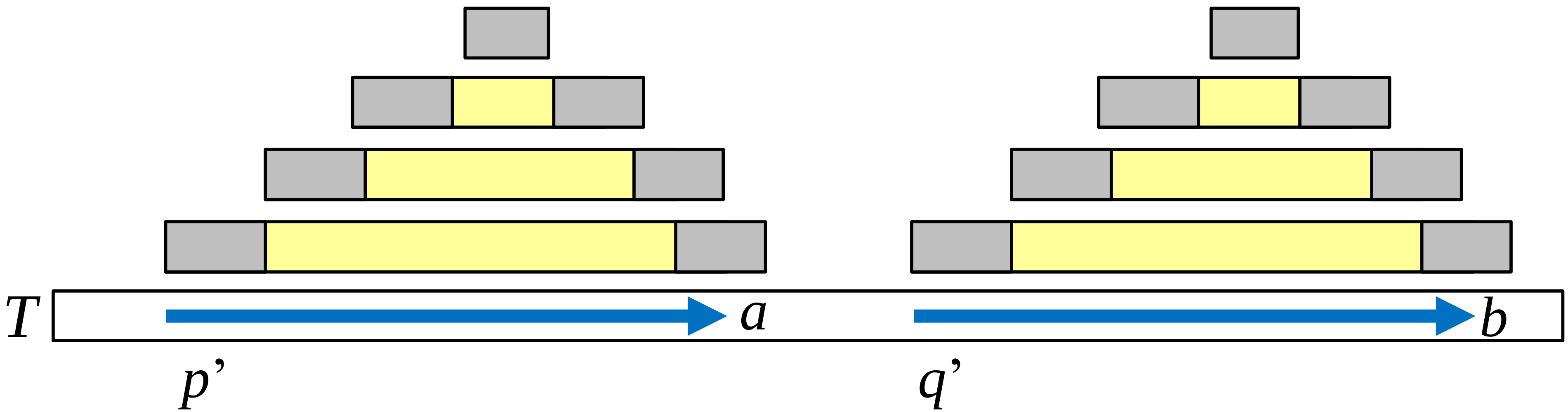
Faster LCE algorithm on SLP

5. By the property of signature encoding, at each level of the signature tree, there must be a <u>common sequence</u> of signatures for $\mathbf{LCE}(p', q')$ (yellow parts).



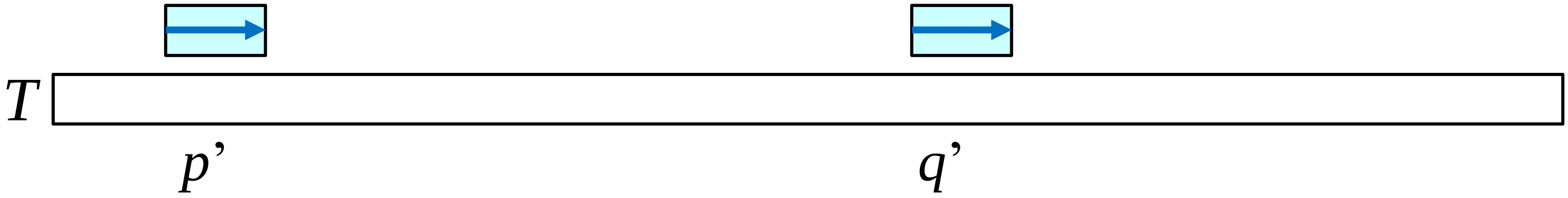
Faster LCE algorithm on SLP

5. [Cont.] The left boundaries of length $\Delta_L + O(1)$ may or may not be equal depending on the left contexts at each level, while the right boundaries of length $\Delta_R + O(1)$ always have a mismatch.



Faster LCE algorithm on SLP

6. In a bottom-up manner, we re-compute the left boundary signatures of length $\Delta_L + O(1)$ ignoring their left contexts, and compare them until we find a mismatch.



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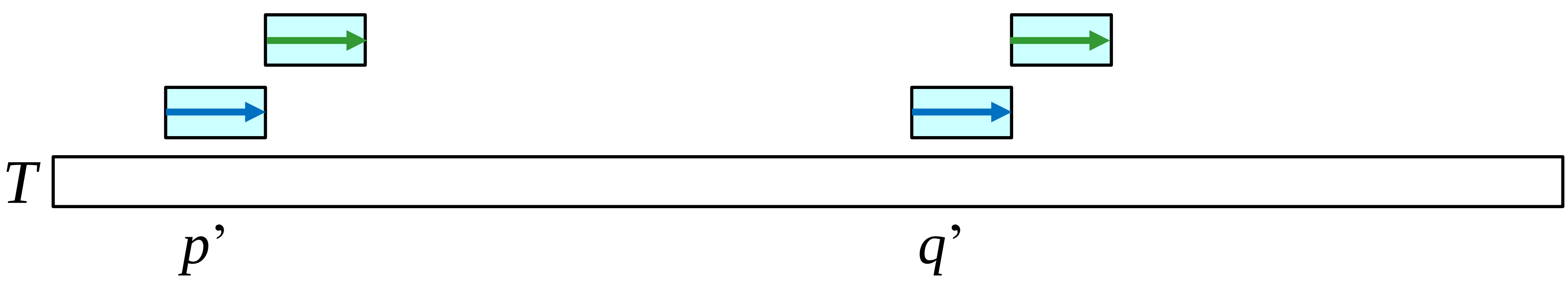

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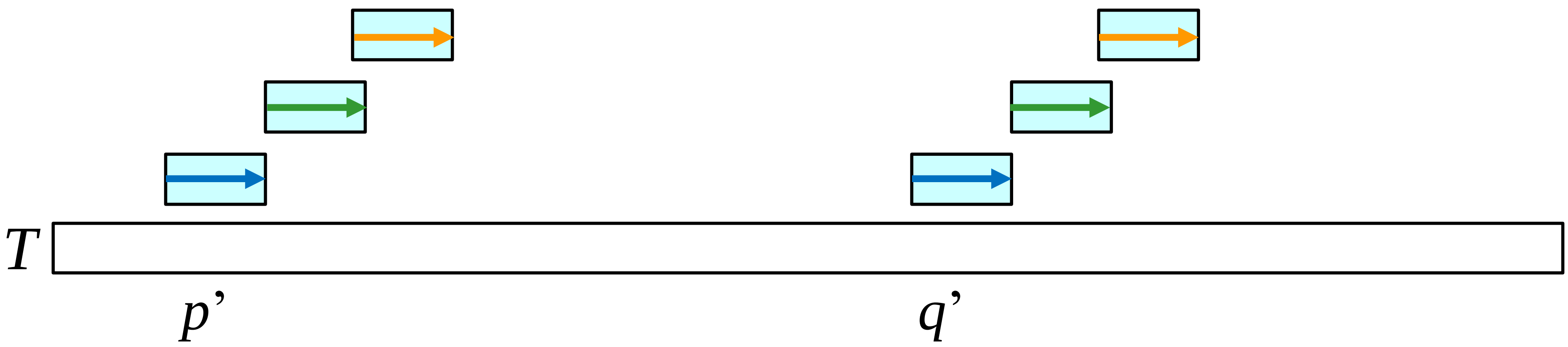

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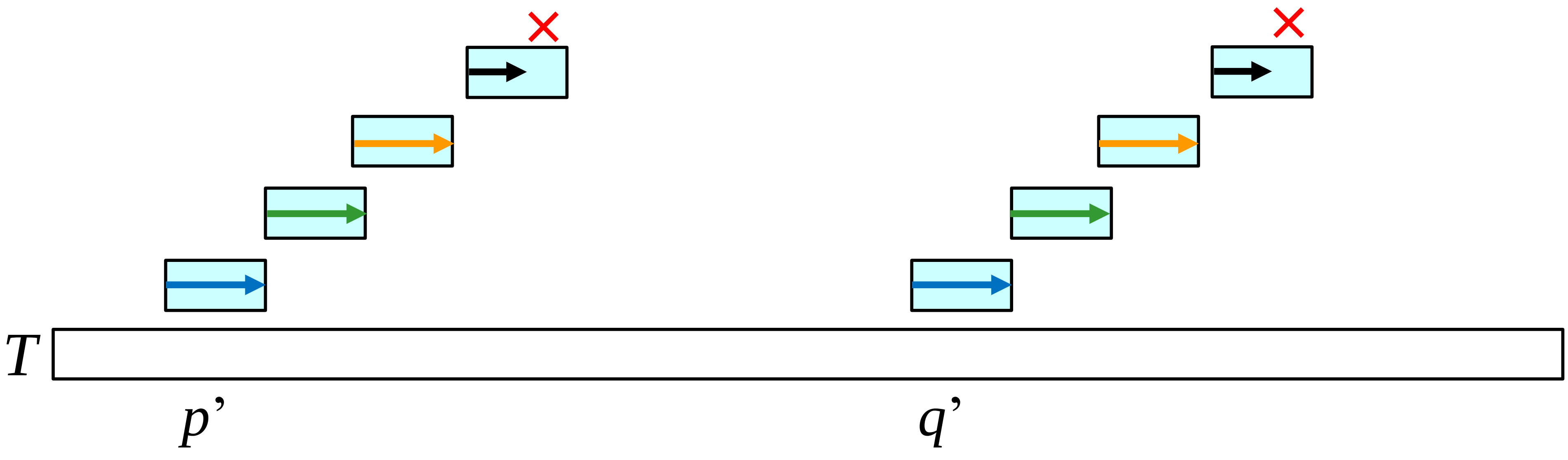
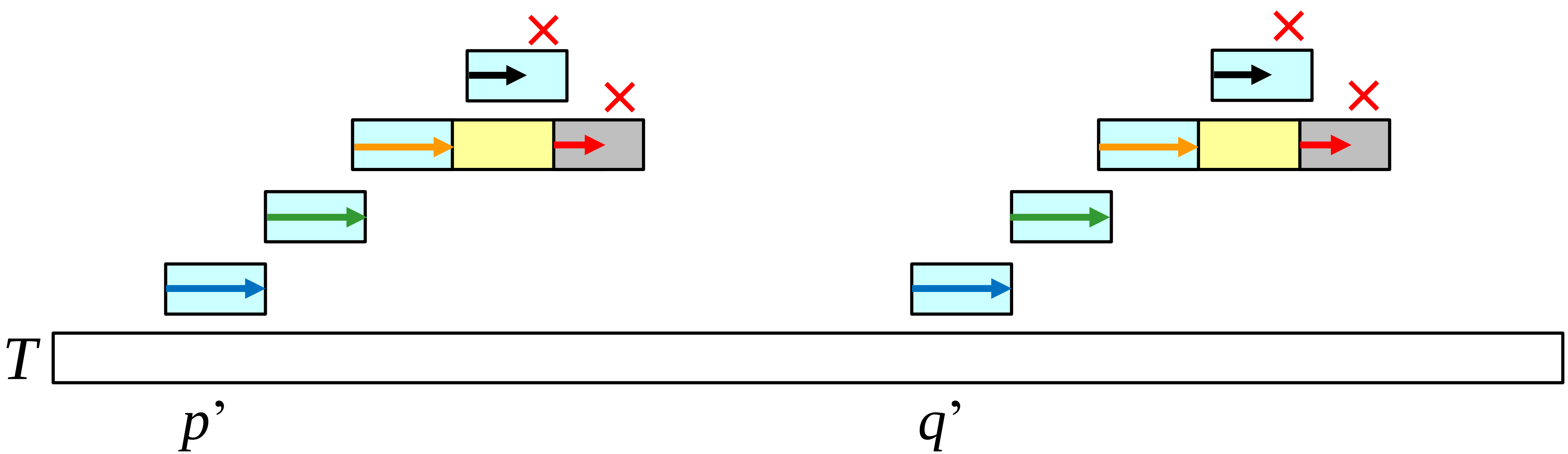


Faster LCE algorithm on SLP

7. In a top-down manner, we compare the right boundary signatures of length $\Delta_R + O(1)$ until we find the first mismatch.



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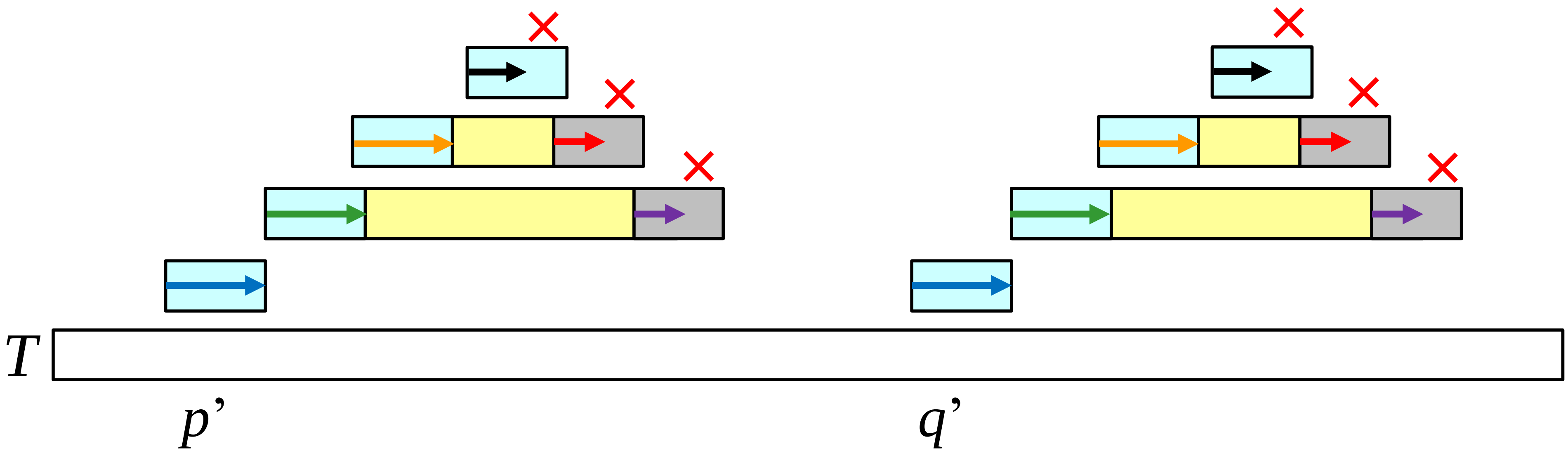

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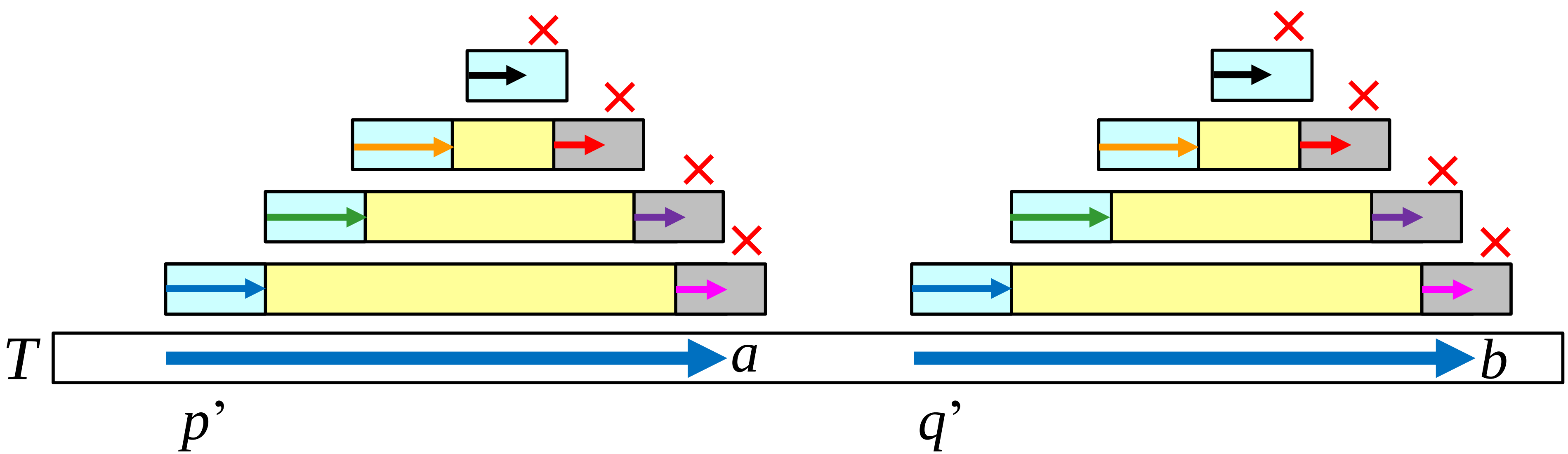


Analysis of LCE query time

- ✓ The paths from the root to the p'th and q'th leaves of the signature tree can be found in $O(\log u)$ time, since its height is $O(\log u)$.
- ✓ The total number of signatures to re-compute and to compare is $O(\log^* u \log L)$, since:
  - ➢ $\Delta_L \leq \log^* u + 6$ and $\Delta_R \leq 4$, and
  - ➢ the first mismatch is found at the $(\log L)$th level from the bottom.
- ✓ Therefore, LCE query can be answered in $O(\log u + \log^* u \log L)$ time.

From SLP to signature encoding

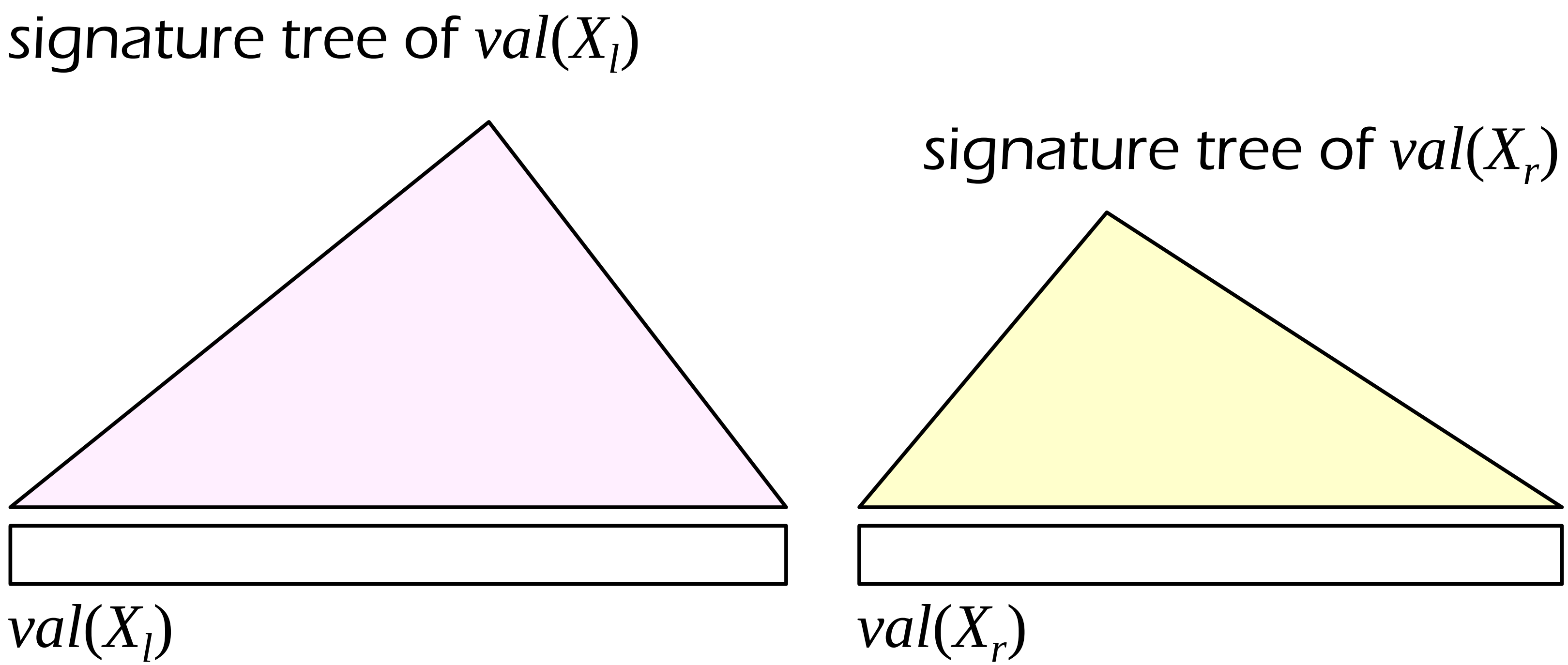
Lemma 3 (SLP to signature encoding)

Given an SLP $S = \{X_i \to expr_i\}_{i=1}^{n}$ of size n which derives a string T of length u, we can compute the signature encoding of T in $O(n \operatorname{loglog} n \log^* u \log u)$ time.

✓ In this talk I show a simpler $O(n \log n \log^* u \log u)$-time construction.

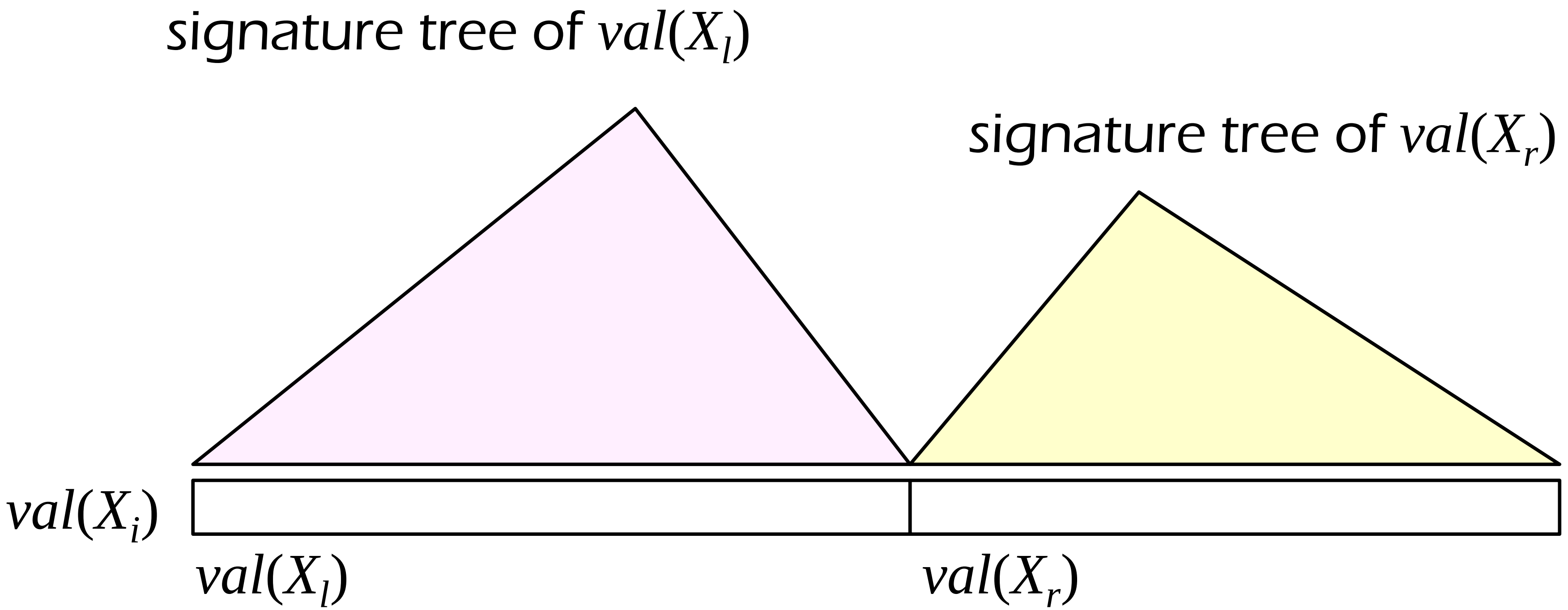
From SLP to signature encoding

- ✓ Assume that, for a production $X_i \rightarrow X_l X_r$, we have computed the signature encodings of the decompressed strings $val(X_l)$ and $val(X_r)$.



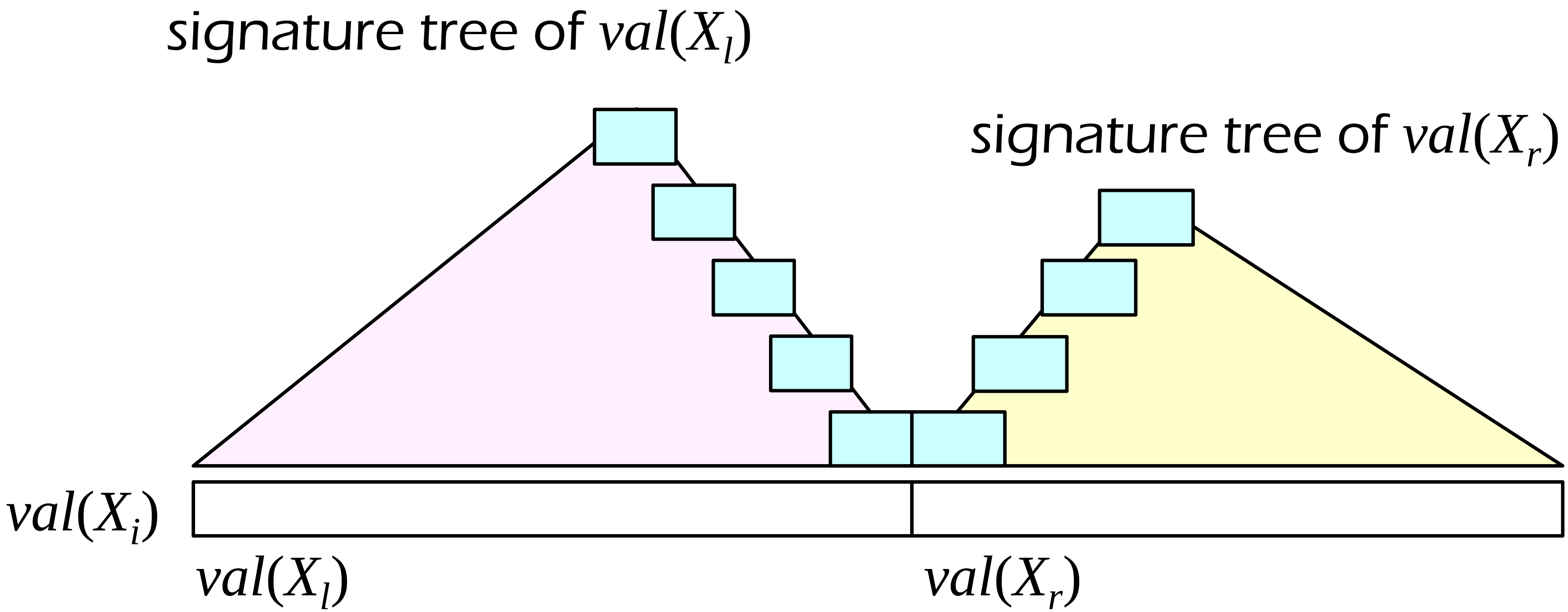
From SLP to signature encoding

✓ By "concatenating" the signature trees of $val(X_l)$ and $val(X_r)$, we obtain the signature tree of $val(X_i)$.



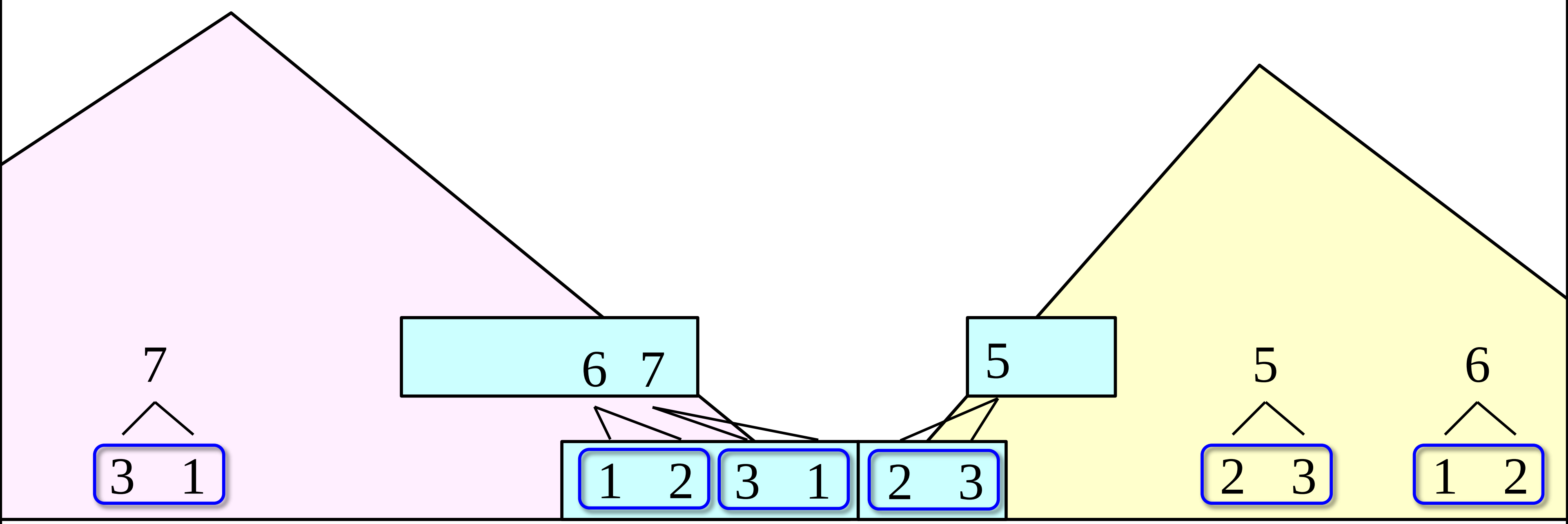
From SLP to signature encoding

- ✓ In a bottom-up manner, we re-compute the boundary signatures of length $\Delta_R+O(1)$ and $\Delta_L+O(1)$ each, and concatenate the new signatures level-wise.



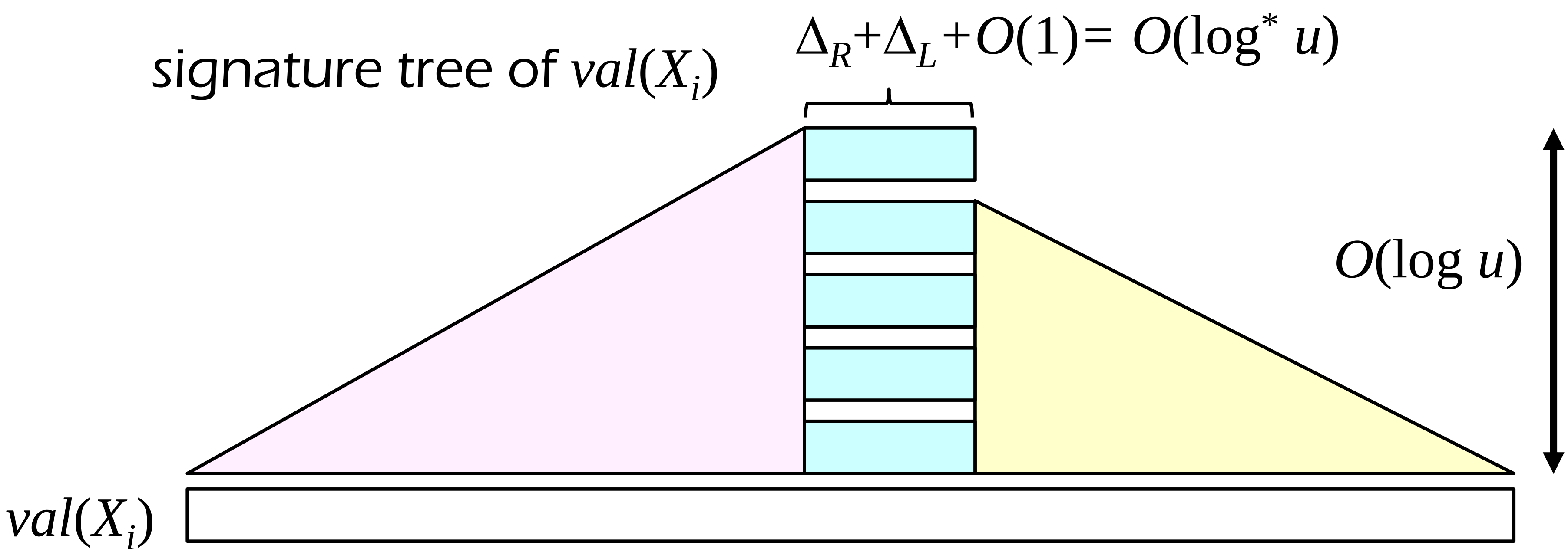
From SLP to signature encoding

- ✓ If a block of re-computed signatures already exists somewhere else, then we assign the same signature to the block at the next level.
  This is done in $O(\log n)$ time each, using a BST.



From SLP to signature encoding

- ✓ Since the height of each signature tree is $O(\log u)$, we can compute the signature encoding of $val(X_i)$ for each X_i in $O(\log n \log^* u \log u)$ time.



How much space?

Lemma 4 [Sahinalp & Vishkin, '95]

The number of signatures involved in the signature encoding of string T of length u is $O(z \log^* u \log u)$, where z is the number of factors in the Lempel-Ziv 77 factorization of T.

✓ In our data structure, we need an additive n term to store beginning positions of occurrences of all non-terminals in the derivation tree of X_n.

Main result

Theorem 1

For any SLP $S = \{X_i \rightarrow expr_i\}_{i=1}^{n}$ of size n which represents a string T of length u, there exists a data structure which

- supports LCE in $O(\log u + \log^* u \log L)$ time;
- requires $O(n + z \log^* u \log u)$ space;
- can be built in $O(n \log\log n \log^* u \log u)$ time,

where L is the LCE length and z is the size of the LZ77 factorization of T.

App 1: Finding palindromes

Problem 2 (finding palindromes on SLP)

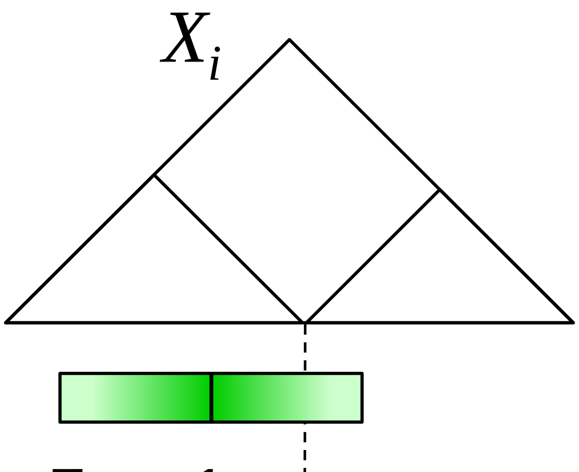
Given an SLP $S = \{X_i \rightarrow expr_i\}_{i=1}^{n}$ representing a string T, compute a compact representation of all maximal palindromes in T.

maximal palindromes

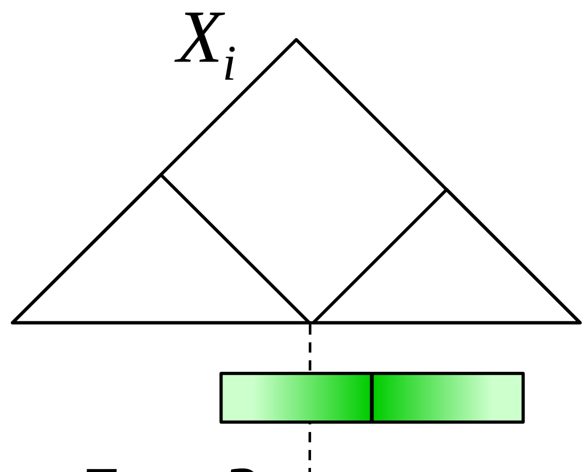
$T =$ a b b b a a b b b b a b b b a a b

Stabbed Palindromes

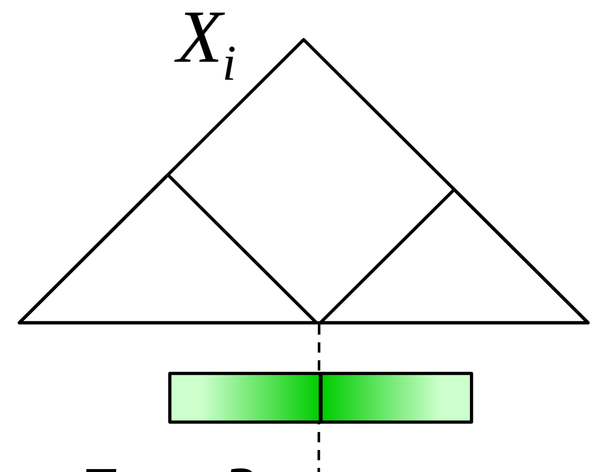
- ✓ For each non-terminal X_i, there are 3 different types of "stabbed" maximal palindromes.



Type 1



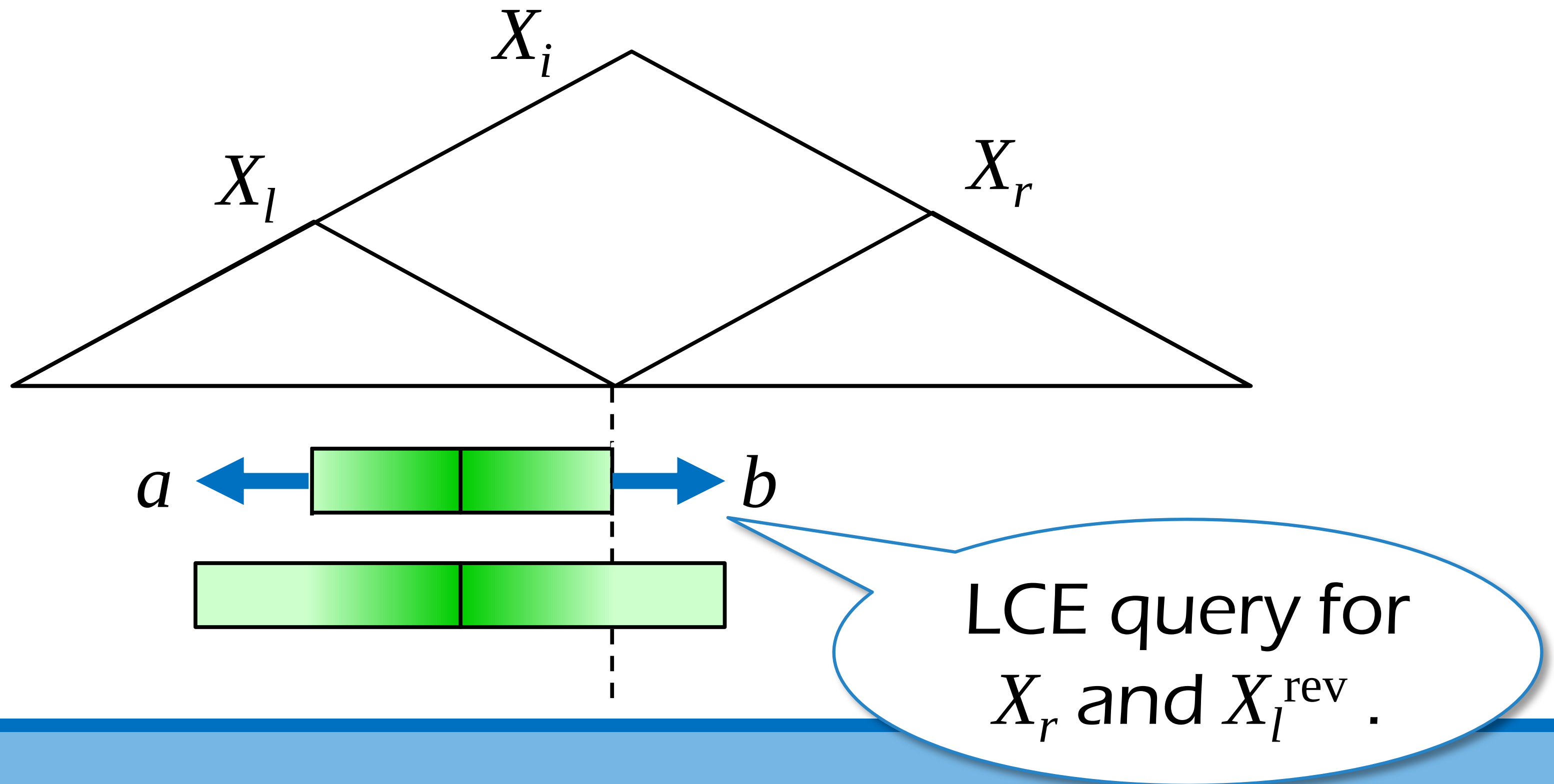
Type 2



Type 3

Computing Type 1 Palindromes

- ✓ Each Type 1 maximal palindrome of X_i can be computed by <u>extending the arms</u> of a suffix palindrome of X_l.



Suffix Palindromes

Lemma 5 [Apostolico et al., '95]

For any string of length k, the lengths of its suffix palindromes can be represented by $O(\log k)$ arithmetic progressions.

✓ We can extend the arms of the suffix palindromes belonging to the same arithmetic progression in a batch, using periodicity.

App 1: Finding Palindromes

Theorem 2

Given an SLP of size n, an $O(n \log u)$-size representation of all maximal palindromes of string T can be computed in $O(n \log^* u \log^2 u)$ time.

With this representation, given an interval $[i, j]$, we can decide whether the substring $T[i..j]$ is a maximal palindrome or not in $O(\log u)$ time.

App 2: Comparing Suffixes on SLP

Problem 3 (lexicographical comparison of suffixes)

Preprocess an input SLP representing string T so that later, any suffixes of the string T can be lexicographically compared efficiently.

App 2: Comparing Suffixes on SLP

Theorem 3

We can preprocess an input SLP of size n representing string T of length u in $O(n \log\log n \log^* u \log u)$ time such that later, any suffixes of T can be lexicographically compared in $O(\log u + \log^* u \log L)$ time, where L is the length of the LCP of the suffixes.

- ✓ Since the height of the signature tree is $O(\log u)$, this theorem is immediate from our LCE data structure.

App 3: Lyndon factorization on SLP

Problem 4 (Lyndon factorization on SLP)

Given an SLP $S = \{X_i \to expr_i\}_{i=1}^{n}$ representing a string T, compute the factor boundaries of the Lyndon factorization of T.

Lyndon word

Definition

A string is said to be a Lyndon word if it is lexicographically smaller than any of its proper cyclic shifts.

For example, “`aaaab`”, “`abc`”, “`bcbcc`” are Lyndon words.

Lyndon factorization

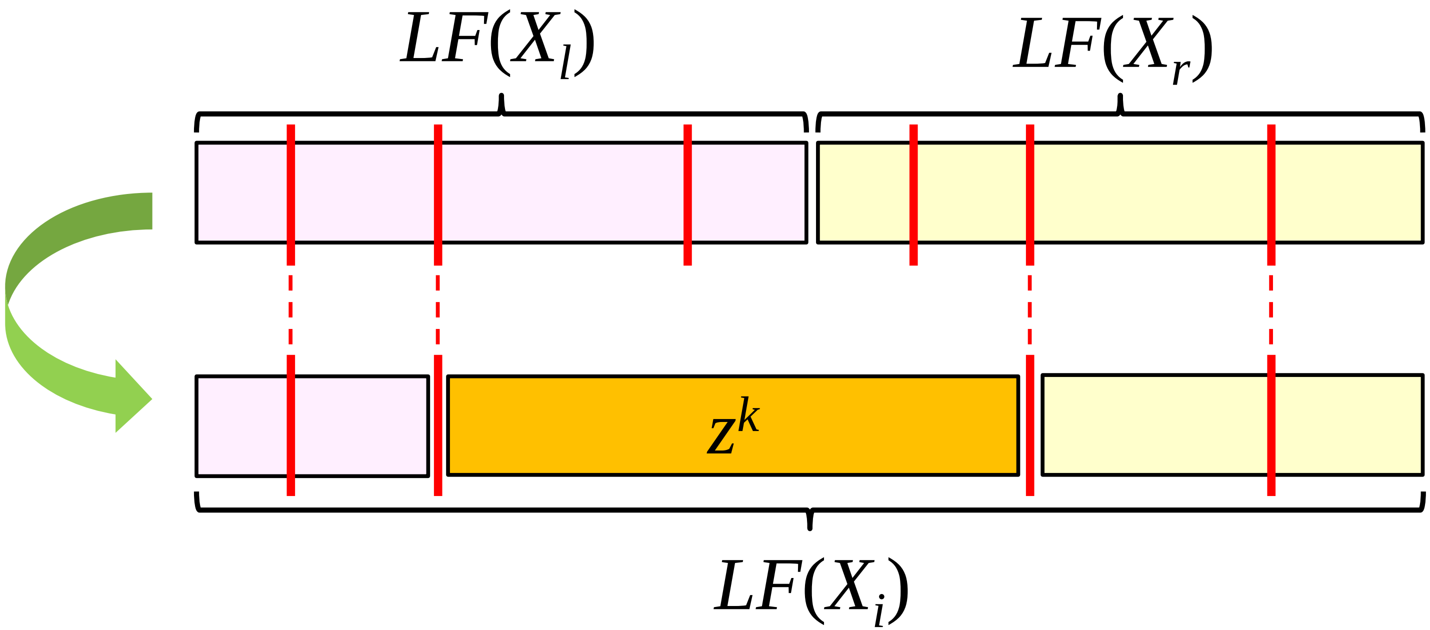
Definition

The Lyndon factorization $LF(T)$ of a string T is the factorization $u_1^{p_1}, \ldots, u_m^{p_m}$ of T such that $u_1, \ldots, u_m$ is a sequence of Lyndon words in lexicographical descending order, and $p_i \geq 1$.

$$T = \underbrace{\text{a b c}}_{u_1} \mid \underbrace{\text{a b b}}_{u_2} \mathrel{\vdots} \underbrace{\text{a b b}}_{u_2} \mid \underbrace{\text{a a b c}}_{u_3} \mid \underbrace{\text{a}}_{u_4} \mathrel{\vdots} \underbrace{\text{a}}_{u_4} \mathrel{\vdots} \underbrace{\text{a}}_{u_4}$$

$$LF(T) = \underbrace{(\text{abc})^1}_{u_1^1} \mid \underbrace{(\text{abb})^2}_{u_2^2} \mid \underbrace{(\text{aabc})^1}_{u_3^1} \mid \underbrace{(\text{a})^3}_{u_4^3}$$

Lyndon factorization on SLP



- ✓ I et al. showed an algorithm which computes $LF(X_i)$ with $X_i \rightarrow X_l X_r$ in the above manner.
- ✓ The beginning and ending positions of the median Lyndon factor z^k can be found by a binary search based on lex-comparison of suffixes.

App 3: Lyndon factorization on SLP

Theorem 4

Given an SLP of size n representing string T of length u, we can compute the factor boundaries of the Lyndon factorization of T in $O(n \log\log n \log^* u \log u)$ time and $O(n^2 + z \log^* u \log u)$ space.

Conclusions & further work

- We proposed a new LCE algorithm on SLPs with $O(\log u + \log^* u \log L)$ query time.
  - ✓ This is the fastest deterministic solution to date.
  - ✓ More details can be found in our arxiv paper: "Dynamic index, LZ factorization, and LCE queries in compressed space".

- Lower bound?
- Other applications?