On expressive power of regular expressions with subroutine calls and lookaround assertions

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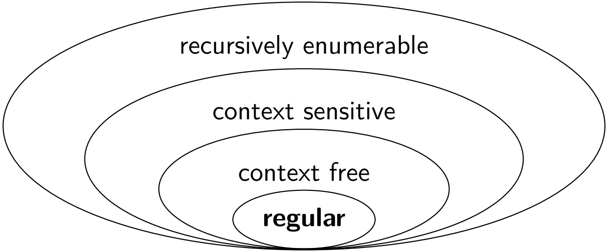
Regular expressions[1] match regular languages[2]

Regular expression is the minimal set of operations to express regular language

$(\texttt{my} + \varepsilon)(\texttt{great})^*\texttt{grandfather}$



[1] Stephen Cole Kleene. "Representation of Events in Nerve Nets and Finite Automata". In: *Automata Studies. (AM-34)*. Ed. by C. E. Shannon and J. McCarthy. Princeton University Press, Dec. 1956, pp. 3–42. ISBN: 978-1-4008-8261-8. DOI: 10.1515/9781400882618-002. (Visited on 08/20/2021).

[2] Noam Chomsky. "On Certain Formal Properties of Grammars". In: *Information and Control* (1959). ISSN: 00199958. DOI: 10.1016/S0019-9958(59)90362-6.

Practical "regular" expressions are different and have various flavours

PCRE2: $\texttt{a}^n\texttt{b}^n$

$(\texttt{a}{\color{red}(?1)}?\texttt{b})$

What language class can be expressed by a particular regex flavour?

?

Expressive power of certain combinations of features was already known

Character class, interval quantifier, concatenation, alternative, iteration: RLs

`[Mm]y ((great){1,3}-grand)(father|mother)`

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Zero-width lookaround assertions: RLs

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Backreferences: some CSLs and not some CFLs

`([a-z]{10})\1`

Formalisation of regex: matching relation[3]

Other formalisms exist

Definition

A matching relation $\rightsquigarrow$ is of the form $(\boldsymbol{r}, \boldsymbol{x}, i) \rightsquigarrow \mathcal{R}$ where $\mathcal{R} = \{i : i \in \mathbb{N} \wedge i \leq |\boldsymbol{x}| + 1\}$ (matching result).

Definition

The language of a regex $\boldsymbol{r} \in \mathbb{E}_{\mathrm{LS},\mathcal{A},\mathcal{X}}$ is $\mathrm{L}(\boldsymbol{r}) = \{\boldsymbol{x} : (\boldsymbol{r}, \boldsymbol{x}, 1) \rightsquigarrow \mathcal{R} \wedge |\boldsymbol{x}| + 1 \in \mathcal{R}\}$.

[3] Nariyoshi Chida and Tachio Terauchi. "On Lookaheads in Regular Expressions with Backreferences". In: *7th International Conference on Formal Structures for Computation and Deduction (FSCD 2022)*. Ed. by Amy P. Felty. Vol. 228. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 15:1–15:18. ISBN: 978-3-95977-233-4. DOI: 10.4230/LIPIcs.FSCD.2022.15.

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Example

$$\frac{a \in \mathcal{A} \land i \leq |\boldsymbol{x}| \land \boldsymbol{x}[i] = a}{(a, \boldsymbol{x}, i) \rightsquigarrow \{i + 1\}}$$

Example (Lookahead)

$$\frac{(\boldsymbol{r}, \boldsymbol{x}, i) \rightsquigarrow \mathcal{R}}{((?{=}\boldsymbol{r}), \boldsymbol{x}, i) \rightsquigarrow \{i \land \mathcal{R} \neq \emptyset\}}$$

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Formalisation of subroutine calls

Definition (Numbered subroutine call)

$$\frac{(\sigma(l), \boldsymbol{x}, i) \rightsquigarrow \mathcal{R}}{((?l), \boldsymbol{x}, i) \rightsquigarrow \mathcal{R}}$$

Definition (Named subroutine call)

$$\frac{(\sigma(\nu(N)), \boldsymbol{x}, i) \rightsquigarrow \mathcal{R}}{((?\mathrm{P}{>}N), \boldsymbol{x}, i) \rightsquigarrow \mathcal{R}}$$

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Example

$$\dfrac{\dfrac{}{((?(\mathrm{DEFINE})(?{<}S{>}\texttt{a}((?1) \mid \varepsilon)\texttt{b}), \texttt{ab}, 1) \leadsto \{1\}} \qquad \dfrac{\dfrac{\dfrac{}{(\texttt{a}, \texttt{ab}, 1) \leadsto \{2\}} \quad \dfrac{\dfrac{\dfrac{\cdots}{(\texttt{a}((?1) \mid \varepsilon)\texttt{b}, \texttt{ab}, 2) \leadsto \emptyset}}{((?1), \texttt{ab}, 2) \leadsto \emptyset} \quad \dfrac{}{(\varepsilon, \texttt{ab}, 2) \leadsto \{2\}}}{(((?1) \mid \varepsilon), \texttt{ab}, 2) \leadsto \{2\}} \quad \dfrac{}{(\texttt{b}, \texttt{ab}, 2) \leadsto \{3\}}}{(\texttt{a}((?1) \mid \varepsilon)\texttt{b}, \texttt{ab}, 1) \leadsto \{3\}}}{((?\mathrm{P}{>}S), \texttt{ab}, 1) \leadsto \{3\}}}{((?(\mathrm{DEFINE})(?{<}S{>}\texttt{a}((?1) \mid \varepsilon)\texttt{b})(?\mathrm{P}{>}S), \texttt{ab}, 1) \leadsto \{3\}}$$

Example (Context-free languages)

$\{\mathtt{a}^n\mathtt{b}^n\}$, $\{\boldsymbol{w}\boldsymbol{w}^{\mathrm{R}}\}$

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Definition (Context-free grammar)

A quadruple $(\mathcal{V}, \mathcal{A}, \mathcal{R}, S)$ where every member of $\mathcal{R}$ is in the form of $N \to \boldsymbol{v}, \boldsymbol{v} \in (\mathcal{A} \cup \mathcal{V})^*$.

Definition (Derivation step in CFG)

If $\boldsymbol{v}_1 \to \boldsymbol{v}_2 \in \mathcal{R}$ then $\boldsymbol{p}\boldsymbol{v}_1\boldsymbol{s} \Rightarrow \boldsymbol{p}\boldsymbol{v}_2\boldsymbol{s}$ is possible.

Definition (Language generated by grammar)

$\{\boldsymbol{x} \in \mathcal{A}^* : S \Rightarrow^* \boldsymbol{x}\}$

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$$(\{S\}, \{\mathtt{a}, \mathtt{b}\}, \{S \to \mathtt{a}S\mathtt{b} \mid \varepsilon\}, S)$$

$$S$$

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Every context-free grammar can be expressed by a regex with subroutine calls

Conversion of a CFG to an equivalent regex

$$(\{S\}, \{\mathtt{a}, \mathtt{b}\}, \{S \to \mathtt{a}S\mathtt{b} \mid \varepsilon\}, S)$$

1. $\boldsymbol{r}_1 = \mathrm{rx}(\mathtt{a}S\mathtt{b}) = \mathtt{a}(\mathrm{?P}{>}S)\mathtt{b}$

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Every regex with concatenation, alternative, and subroutine call can be expressed by a context-free grammar

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- $\mathfrak{G}_\varepsilon = (\{S_\varepsilon\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{S_\varepsilon \to \varepsilon\}, S_\varepsilon\})$, $\mathfrak{G}_a = (\{S_\mathtt{a}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{S_\mathtt{a} \to \mathtt{a}\}, S_\mathtt{a})\ldots$

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- $\mathfrak{G}_{(?\mathrm{P}>N)} = (\{S_{(?\mathrm{P}>N)}, N\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{S_{(?\mathrm{P}>N)} \to N\}, S_{(?\mathrm{P}>N)})$

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- . . .

Thus it is shown that subroutine calls match exactly context-free languages

Theorem

$\mathbb{L}_{\mathrm{E_S}} = \mathbb{L}_{\mathrm{CF}}$

Adding lookahead to a regex with subroutine calls extends it expressive power

Theorem

$\mathbb{L}_{\mathrm{E_S}} \subsetneq \mathbb{L}_{\mathrm{E_{LS}}}$

Proof.

$$\{\mathtt{a}^g\mathtt{b}^g\mathtt{c}^g : g \in \mathbb{N}\} =$$

$$\mathrm{L}((?{=}(?{<}N_1{>}\mathtt{a}(\varepsilon \mid (?\mathrm{P}{>}N_1))\mathtt{b})\,\mathtt{c})\mathtt{aa}^*(?{<}N_2{>}\mathtt{b}(\varepsilon \mid (?\mathrm{P}{>}N_2))\mathtt{c}))$$

□

Summary

- expressive power of expressions with concatenation, alternative, and subroutine call is equivalent to the class of context-free languages
  - algorithms to convert between regex and context-free grammar
- expressive power of expressions with concatentation, alternative, subroutine call, and lookahead is beyond context-free languages

Future work: expressive power of regex with both subroutine call and lookaround.