Weighted Automata and Weighted Logics with Discounting

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- **Problem:** $weight(P_w) = ?$

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- *Each endomorphism of $\mathbb{R}_{\max}$ is of the above form* (Droste and Kuske '06)

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- **Example** $a \in A$, $k \in \mathbb{R}_{\max}$, Büchi automaton $\mathcal{A} = (\{q\}, in, wt, \{q\})$
- $in(q) = 0$
- $wt(q, a, q) = k,\ wt(q, a', q) = 0\ (a' \in A, a' \neq a)$

- $In^Q(P_w) = \{q \in Q \mid \exists^\omega i : t_i = (q, a_i, q_{i+1})\}$
- P_w is *successful* if $\begin{cases} In^Q(P_w) \cap F \neq \emptyset & \text{Büchi} \\ In^Q(P_w) \in \mathcal{F} & \text{Muller} \end{cases}$
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- *Let* $L \in \omega - Rec(A)$. *Then* $1_L \in \mathbb{R}_{\max}^{p-\omega-rec}\langle\langle A^\omega\rangle\rangle$.

Weighted MSO logic with discounting

Definition

The syntax of the weighted MSO-formulas over A and $\mathbb{R}_{\max}$ is given by

$$\varphi := k \mid P_a(x) \mid \neg P_a(x) \mid S(x,y) \mid \neg S(x,y) \mid x \leq y \mid \neg(x \leq y) \mid x \in X \mid \neg(x \in X) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x \,.\, \varphi \mid \exists X \,.\, \varphi \mid \forall x \,.\, \varphi$$

$k \in \mathbb{R}_{\max}$, $a \in A$.

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Semantics of $\varphi \in MSO(\mathbb{R}_{\max}, A)$: $\|\varphi\| : A_{\varphi}^{\omega} \to \mathbb{R}_{\max}$.
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- $(\|\neg\varphi\|, (w,\sigma)) = \begin{cases} 0 & \text{if } (\|\varphi\|, (w,\sigma)) = -\infty \\ -\infty & \text{if } (\|\varphi\|, (w,\sigma)) = 0 \end{cases}$, provided that φ is of the form $P_a(x)$, $S(x,y)$, $x \leq y$ or $x \in X$

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- **Example** Let $\varphi = \forall x \centerdot (P_a(x) \to k)$, $k \in \mathbb{R}_{\max}$.

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- **Example** Let $\varphi = \forall x \,.\, (P_a(x) \to k)$, $k \in \mathbb{R}_{\max}$.
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  returns the *discounted cost* of a's in w.

Definition

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- Purely syntactic definition!
- $\mathbb{R}_{\max}^{p-aemso} \langle\langle A^\omega \rangle\rangle$: all series in $\mathbb{R}_{\max} \langle\langle A^\omega \rangle\rangle$ which are definable by some almost existential sentence in $MSO(\mathbb{R}_{\max}, A)$

Theorem (second main result)

$$\mathbb{R}_{\max}^{p-\omega-rec}\langle\langle A^{\omega}\rangle\rangle = \mathbb{R}_{\max}^{p-aemso}\langle\langle A^{\omega}\rangle\rangle$$

Proof.

- By induction on the structure of weighted MSO-formulas we show $\mathbb{R}_{\max}^{p-aemso}\langle\langle A^\omega\rangle\rangle \subseteq \mathbb{R}_{\max}^{p-\omega-rec}\langle\langle A^\omega\rangle\rangle$.

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□

- In the paper we have shown corresponding results for finitary series (over finite words).

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- the semiring of polynomials $(K[X], +, \cdot, 0, 1)$ over a variable X and an additively locally finite semiring K.

Corollary

Let K be a computable, additively locally finite, commutative semiring, or let $K = \mathbb{R}_{\max}$ or $K = \mathbb{R}_{\min}$. Let $0 \leq p < 1$. Given an almost existential MSO(K, A)-formula φ whose atomic entries from K are effectively given, we can effectively compute a weighted automaton, resp. a weighted Muller automaton, $\mathcal{A}$ such that $\|\varphi\| = \|\mathcal{A}\|$.

Other work

Weighted automata and weighted logics on

- finite words

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Thank you