A Measure for The Degree of Nondeterminism of Context-Free Languages[1]

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An analysis of a sentence

- it is very complex and has several phases

  a dictionary lookup
  morphological analysis
  disambiguation, ...
  } additional information is inserted into the input sentence – auxiliary symbols

  syntactic analysis
- syntactic analysis
  - often nondeterministic – high complexity
  - "make it as efficient (deterministic) as possible"
  - many different models are used – among others restarting automata
- Questions:
  1. How many symbols must be added in order to get a subsequent analysis deterministic?
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Our model

an artificial example:

The language $L_e = \{ww^R \mid w \in \{a, b\}^*\}$ is not deterministic context-free, i.e. deterministic for a pushdown automaton, but an insertion of a single symbol in the middle of an input word – $w\#w^R$ – makes its parsing simple.

- categories are added and disambiguated by a nondeterministic process followed by
- a deterministic analysis by reduction modelled by a restarting automaton

Outline

1. Restarting automaton
   - Definition
   - Meta-instructions
   - Languages defined by restarting automata
2. Lexicalized restarting automata
   - Deterministic and lexicalized restarting automata
   - Monotone restarting automata
   - How to measure nondeterminism
   - Word-alphabet expansion hierarchy
3. Conclusions

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a tool for modeling **analysis by reduction**

Martin, Peter and Jane work very slowly.

Peter and Jane work very slowly.

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- restarting automaton consists of:
  - finite control
  - read/write window of fixed size
  - operations: move right, rewrite, restart, accept

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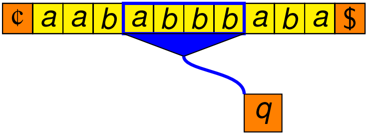

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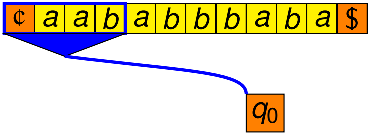

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  - starts on the left end
  - consists of cycles = parts between two restarts
  - in one cycle exactly one rewriting of the content of its window must occur (local change), it must shorten the tape
- this model emphasized locality of one reduction step
- even for CFL recognition there are needed additional (non-input) working symbols

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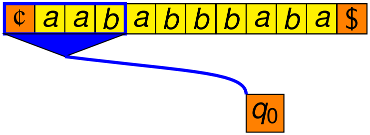

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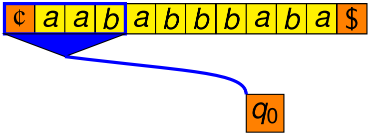

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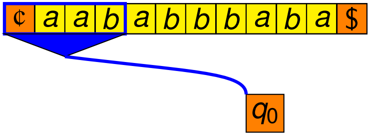

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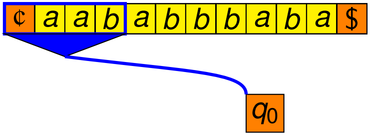

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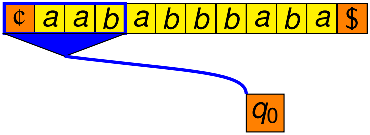

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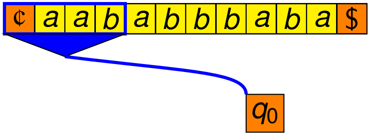

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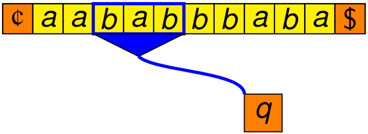
briefly RRWW-automaton



$M = (Q, \Sigma, \Gamma, ¢, \$, q_0, k, \delta)$:

- Q is a finite set of states,
- Σ is a finite input alphabet,
- Γ is a finite tape alphabet, $\Sigma \subseteq \Gamma$,
- $¢, \$$ are sentinels, $\{¢, \$\} \cap \Sigma = \emptyset$
- $q_0 \in Q$ is the initial state
- δ is the transition relation = a finite set of instructions.

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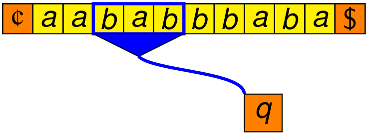

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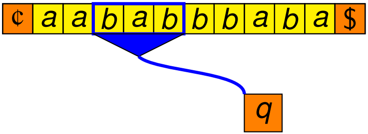

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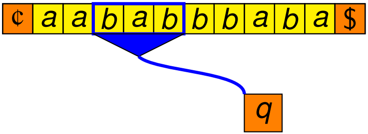

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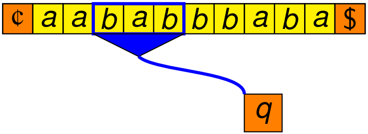

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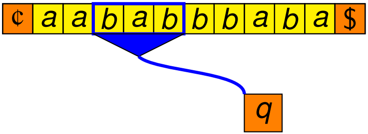

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Computation of a restarting automaton

Cycles, tails

a cycle the part of an computation between two restarting configurations

a tail the part of an computation after the last restart configuration and halt (accepting or rejecting).

$w_1 \vdash^c_M w_2 \vdash^c_M \cdots \vdash^c_M w_{n-1} \vdash^c_M w_n$

Note: if w_n is accepted by M then all $w_1, \ldots, w_n$ are accepted by M.

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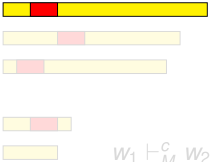

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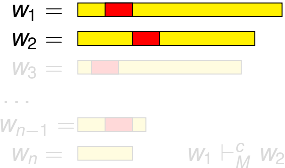

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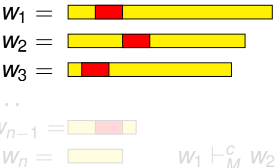

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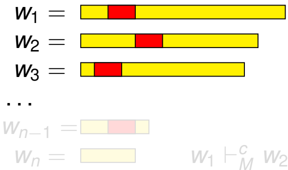

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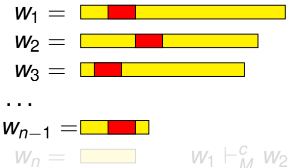

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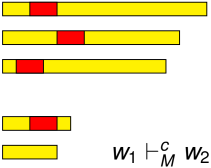

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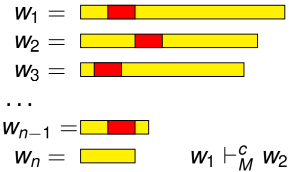

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Example

- A restarting automaton for the language

$$L'_e = \{w\#w^R | w \in \{a, b\}^*\}$$

- Sample computation on the word $abb\#bba$:
  - $¢abb\#bba\$$
  - $¢ab\#ba\$$
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  - $¢\#\$$
  - Accept
- It can be done by a deterministic restarting automaton.

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Meta-instructions

A more convenient representation

$(E_1, u \to v, E_2)$ a rewriting meta-instruction,

- $E_1, E_2 \subseteq \Gamma^*$ are regular languages called constraints
- $u, v \in \Gamma^*$ such that $|u| > |v|$,
- if $w = w_1 u w_2$, where $¢ \cdot w_1 \in E_1$, $w_2 \cdot \$ \in E_2$ then

$$w = \boxed{w_1}\boxed{u}\boxed{w_2}$$

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Languages defined by a RRWW-automaton

- A word w is accepted by M if there exists a computation which starts by the initial configuration $q_0¢w\$$ and ends by an accepting configuration.
- The set of *all* words accepted by M is denoted as $L_C(M)$ and it is called the characteristic language accepted by the RRWW-automaton M.
- $L(M) = L_C(M) \cap \Sigma^*$ denotes the *input language* accepted by M
- By Pr^Σ we denote the projection from Γ^* onto Σ^* which leaves out all the auxiliary symbols.
- If $v = \mathrm{Pr}^\Sigma(w)$ we say that w is an expanded version of v.
- $L_P(M) := \mathrm{Pr}^\Sigma(L_C(M))$ denotes the proper language of M – a word $v \in \Sigma^*$ belongs to $L_P(M)$ if and only if v has an expanded version u such that $u \in L_C(M)$.

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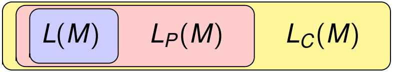

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M, with input alphabet $\Sigma = \{a, b\}$, one auxiliary symbol C ($\Gamma = \Sigma \cup \{C\}$) and the following meta-instructions:

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An RRWW-automaton $M = (Q, \Sigma, \Gamma, ¢, \$, q_0, k, \delta)$ is

- deterministic if its transition relation is deterministic (i.e. after transforming the meta-instructions into a low level instructions we get a deterministic automaton).

Fact

If M is a deterministic RRWW*-automaton, then the membership problem for the language $L_C(M)$ is solvable in linear time.*

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- monotone computation – the places of rewriting in cycles do not increase their distance from the right sentinel. Tails are not considered here.

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CFL = $\mathcal{L}_P$(mon-lex-RRWW).

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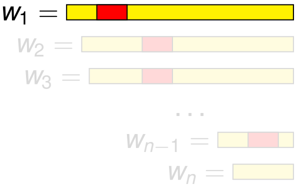

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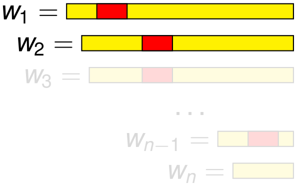

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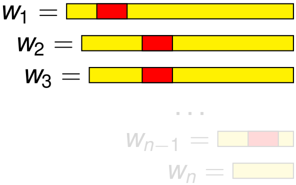

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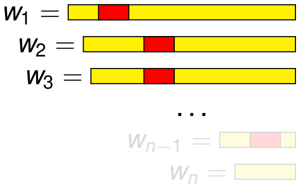

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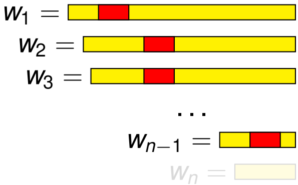

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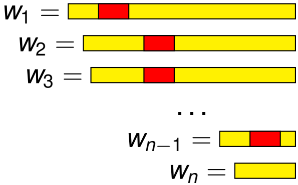

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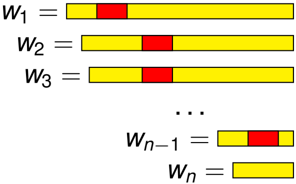

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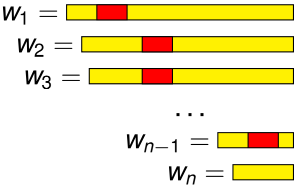

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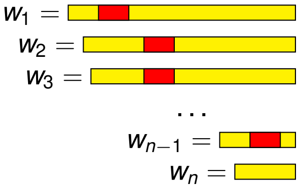

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Word and word-alphabet expansion

- M has word-expansion m (denoted by $\mathsf{W}(m)$-) if each word from $L_C(M)$ contains at most m occurrences of auxiliary symbols.
- M has word-alphabet-expansion (m, j) (denoted by $\mathsf{WA}(m, j)$-) if M has word-expansion m, and Γ contains at most j different auxiliary symbols ($|\Gamma \setminus \Sigma| \leq j$).

Theorem

If M is a $\mathsf{W}(m)$-RRWW-automaton for some constant $m \geq 0$, then the membership problem for the language $L_P(M)$ is solvable in deterministic polynomial time.

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DCFL $= \mathcal{L}_{\mathrm{P}}$(W(0)-mon-RRWW) $= \mathcal{L}_C$(W(0)-mon-RRWW).

- let $L_3 := \bigcup_{1 \leq k \leq 3} \{ a^n(b^k)^n \mid n \geq 0 \}$ over $\Sigma_0 := \{a, b\}$.

Lemma

$L_3 \in \mathcal{L}_{\mathrm{P}}(\text{WA}(1,1)\text{-mon-RRWW}) \setminus \mathcal{L}_P(\text{W}(0)\text{-RRWW})$.

- L_3 is the proper language of the automaton with the following meta-instructions (C is an auxiliary symbol):

(1)	$(¢ \cdot a^*, ab \to \lambda, b^* \cdot \$)$,	(3)	$(¢ \cdot aCa^*, abbb \to \lambda, b^+ \cdot \$)$,
(2)	$(¢ \cdot Ca^*, abb \to \lambda, b^* \cdot \$)$,	(4)	$(¢ \cdot (C + aCbbb + \lambda) \cdot \$, \text{Accept})$.

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DCFL $= \mathcal{L}_{\mathrm{P}}$(W(0)-mon-RRWW) $= \mathcal{L}_C$(W(0)-mon-RRWW).

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- Let L_{pal} denote the language of palindromes $L_{\mathrm{pal}} := \{\, w \in \Sigma_0^* \mid w = w^R \,\}$.

Lemma

L_{pal} belongs to the class $\mathcal{L}_{\mathrm{P}}(\mathrm{WA}(1,1)\text{-mon-RRWW})$, but it is not contained in the class $\mathcal{L}_{\mathrm{P}}(\mathrm{W}(0)\text{-RRWW})$.

- For $j \in \mathbb{N}_+$, let $L_j := \bigcup_{1 \le i \le j} \{\, a^n (b^i)^n \mid n \ge 0 \,\}$, and

$$L_p(1,j) := \{\, ucvcw \mid uw \in L_{\mathrm{pal}},\ |u| \ge |w| \ge 0, \text{ and } v \in L_j \}.$$

Based on the construction of M_{pal} we can construct a monotone RRWW-automaton $M^p_{1,j}$ that has word-alphabet-expansion $(1,j)$ and that satisfies $L_{\mathrm{P}}(M^p_{1,j}) = L_p(1,j)$.

Lemma

For all $j \in \mathbb{N}_+$, $L_p(1,j) \in \mathcal{L}_{\mathrm{P}}(\mathrm{WA}(1,j)\text{-mon-RRWW})$, while for $j > 1$, $L_p(1,j) \notin \mathcal{L}_{\mathrm{P}}(\mathrm{WA}(1,j-1)\text{-RRWW})$.

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Now, for all $m, j \in \mathbb{N}_+$, let $L_p(m,j) := L_p(1,j) \cdot (\{d\} \cdot L_{\mathrm{pal}})^{m-1}$.

Lemma

(a) *For all* $m, j \geq 1$, $L_p(m,j) \in \mathcal{L}_{\mathrm{P}}(\mathrm{WA}(m,j)\text{-mon-RRWW})$.

(b) *For all* $m, j \geq 1$, $L_p(m,j) \notin \mathcal{L}_{\mathrm{P}}(\mathrm{W}(m-1)\text{-RRWW})$.

(c) *For all* $m \geq 1$ *and* $j \geq 2$,
$L_p(m,j) \notin \mathcal{L}_{\mathrm{P}}(\mathrm{WA}(m,j-1)\text{-RRWW})$.

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In processing a word from $L_p(m,j)$, a lexicalized RRWW-automaton M' must be able to distinguish between the j possible values for the parameter i in the factor $c \cdot a^n \cdot b^{i \cdot n} \cdot c$ inserted within the first of the above palindromes. As it scans its tape strictly from left to right, it only has the first of the above-mentioned auxiliary symbols to encode the correct value. It follows that j different auxiliary symbols must be available to M'.

Word-alphabet expansion hierarchy

Theorem

For all $\mathrm{X} \in \{\mathrm{RRWW}, \text{mon-RRWW}\}$*, we have the following proper inclusions:*

(a) $\mathcal{L}_P(\mathrm{W}(m)\text{-X}) \subset \mathcal{L}_P(\mathrm{W}(m+1)\text{-X})$ *for all* $m \geq 0$.

(b) $\mathcal{L}_P(\mathrm{WA}(m,j)\text{-X}) \subset \mathcal{L}_P(\mathrm{WA}(m+1,j)\text{-X})$ *for all* $m \geq 0, j \geq 1$.

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Corollary

$\bigcup_{m\geq 0} \mathcal{L}_P(\mathrm{W}(m)\text{-mon-RRWW}) \subset \mathcal{L}_P(\text{lex-mon-RRWW}) = \mathrm{CFL}$.

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Conclusions

- The degree of word-expansion and the word-alphabet-expansion are new measures for the degree of nondeterminism for proper languages of restarting automata.
- we have obtained infinite hierarchies of language classes for monotone and for non-monotone RRWW-automata that are lexicalized.
- In the monotone case these classes form an infinite hierarchy between DCFL and CFL.
- Moreover, for each finite degree m of word-expansion, the number of available different auxiliary symbols yields an infinite hierarchy within $\mathcal{L}_{\mathrm{P}}(\mathrm{W}(m)\text{(-mon)-RRWW})$.
- Any lexicalized RRWW-automaton has word-expansion that is bounded from above by a linear function. It remains to study those languages that are obtained as proper languages of lexicalized RRWW-automata for which the word-expansion is non-constant, but bounded from above by a slowly growing function.

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