Making de Bruijn Graphs Eulerian

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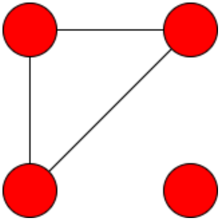
Outline

Graphs

A **graph** G consists of

- a finite set of nodes V
- a finite set of edges E connecting the nodes



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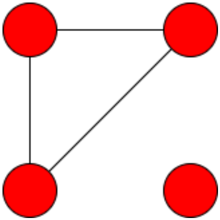

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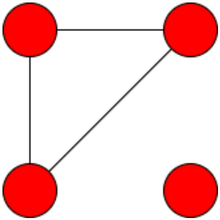

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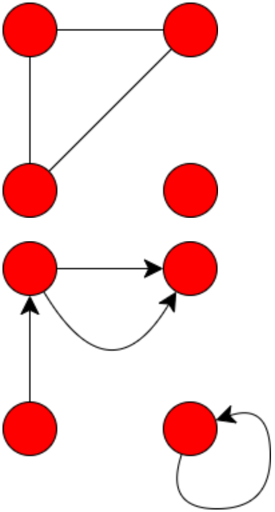

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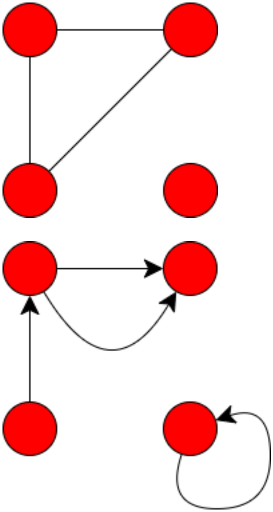
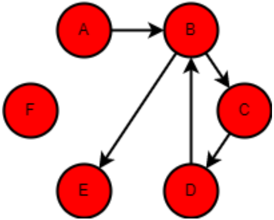


Our problem

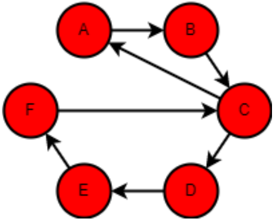
We work with directed multigraphs.

More Graph Definitions

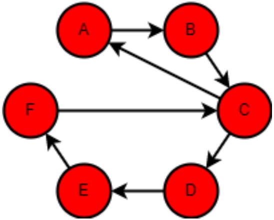
Walk
Sequence of nodes connected by edges



Circuit
Walk with the same first and last node



Graph Problems



Eulerian walk/ciruit

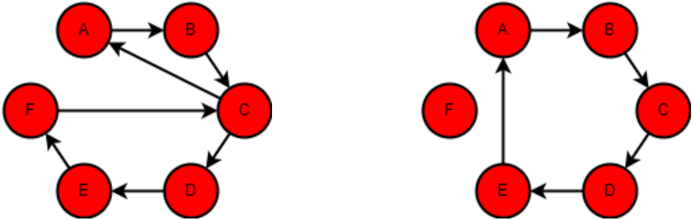
A walk/circuit which visits every edge exactly once.

Euler's Theorem

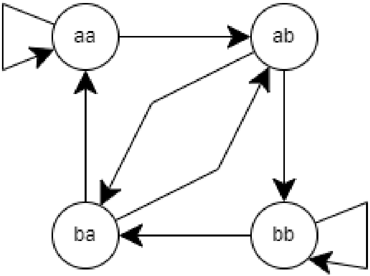
Theorem

A graph contains a Eulerian circuit if and only if

- *the edges are connected and*
- *the nodes are balanced.*



De Bruijn Graphs



Complete order 3 de Bruijn graph
over alphabet $\Sigma = \{a, b\}$

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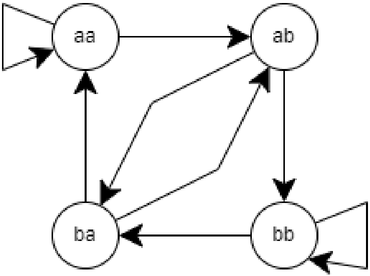

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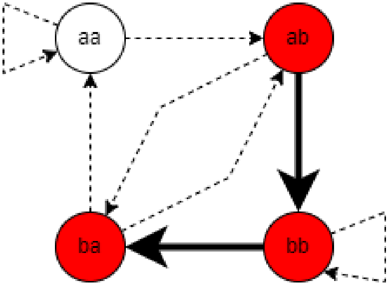

Order 3 de Bruijn graph of *abba*

Outline

Problem Definition

Problem (Eulerian Extension)

We are given a multigraph $G = (V, E)$ *where* $V \subseteq \mathcal{V}$ *and a set of forbidden edges* $F \subseteq \mathcal{V} \times \mathcal{V}$*. Find a minimum multiset of feasible edges* $A \subseteq (\mathcal{V} \times \mathcal{V}) \setminus F$ *and a set of nodes* $B \subseteq \mathcal{V}$ *such that*

- $(V \cup B, E \cup A)$ *is connected and*
- $(V \cup B, E \cup A)$ *is balanced.*

CWI

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$G = (V, E)$ is a de Bruijn graph of strings

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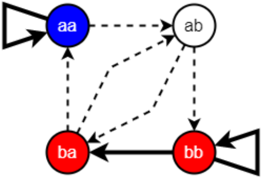
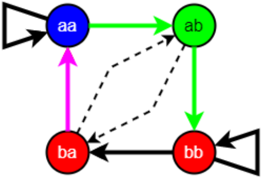
Extend-DBG $\mathcal{V} = \Sigma^{k-1}$ and F is all edges not in the complete de Bruijn graph.





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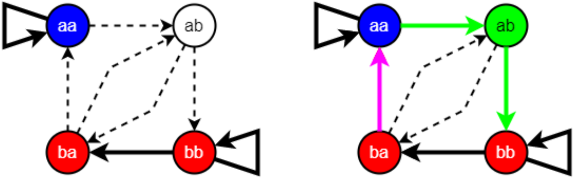
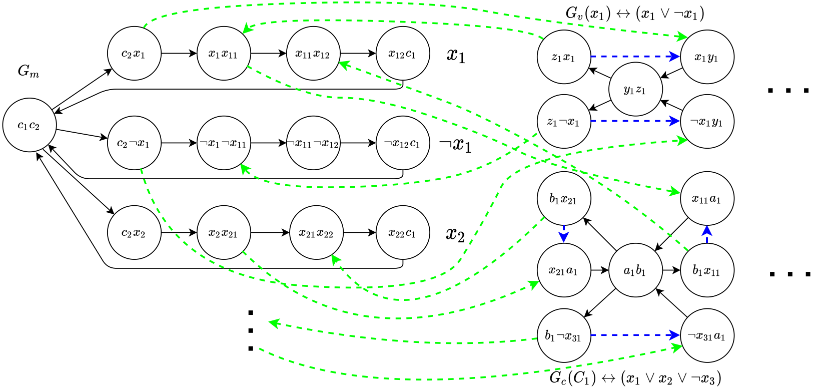


R-Extend-DBG $\mathcal{V} = V$ and F is all edges not in the complete de Bruijn graph.

Outline

Theorem

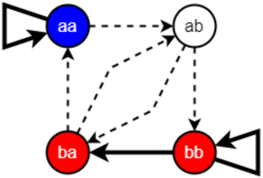
Restricted Eulerian Extension is NP-hard (even if the graph is a de Bruijn graph).



Outline

Connect and Balance (CAB)

Idea



Connect and Balance (CAB)

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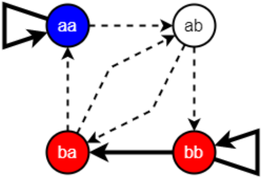
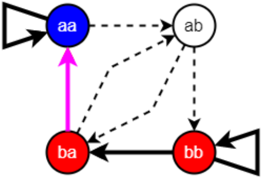
1. Connect the graph





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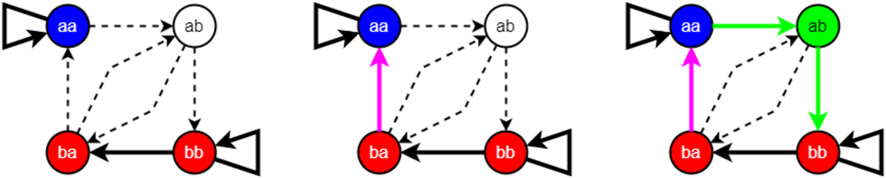
2. Balance the graph



CWI

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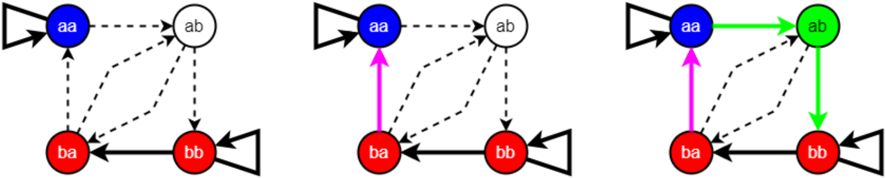

Remark

We do not solve the Eulerian Extension problem optimally.

Connecting

Kruskal's algorithm:

- Connect closest components
- Optimal for minimum spanning tree

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Use graph algorithms.

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Use string algorithms to find the minimum distance between all pairs of nodes in $G = (V, E)$.

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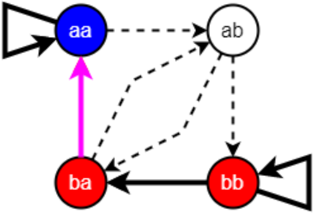
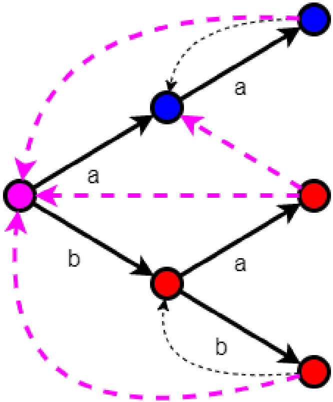
Idea 3

Use an automaton to simultaneously compute all overlaps and than go through them from longest to shortest.

+ $O(k|V|\log|V| + |E|)$

Aho-Corasick Automaton





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- $Z^+ = \{v \mid d^+(v) > d^-(v)\}$
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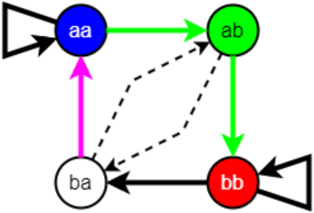

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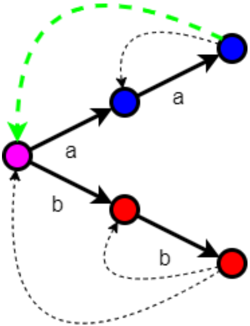

Time Complexity:
$O(|V|k + |E|)$

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Results

Algorithm: CAB (connect and balance)

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Benchmarks:

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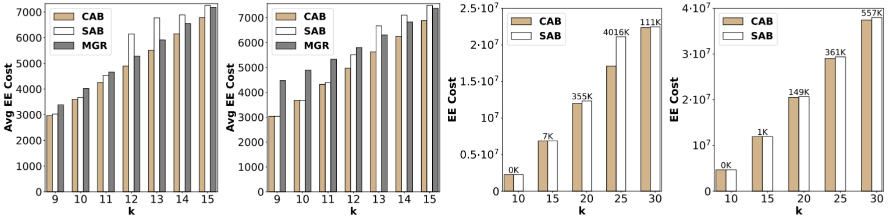

(a) STA samples (b) RHO samples (c) STA (d) RHO