Compression by Contracting Straight-Line Programs

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Algorithms on Compressed Data

Two goals in data compression:

- Store data in a compact form (lossless).
- Support efficient queries directly on the compressed representation.
  $\rightarrow$ Avoid decompression!

compression — algorithmics

Kolmogorov complexity | LZ77 | grammar-based compression | RLE | uncompressed

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A **straight-line program (SLP)** is a context-free grammar $\mathcal{G}$ which produces exactly one string.

Every variable occurs exactly once on the left-hand side of a rule and the variables are topologically ordered.

grammar

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\begin{aligned}
S_4 &\to S_3\, S_2 \\
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derivation tree

The string length is denoted by $N \leqslant 2^{\mathcal{O}(|\mathcal{G}|)}$.

Chomsky normal form: rules of the form $A \to BC$ or $A \to \mathtt{a}$.

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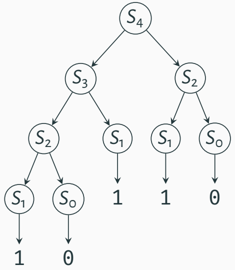

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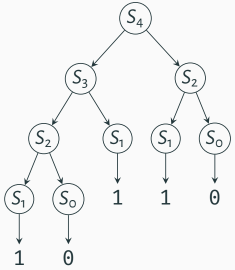

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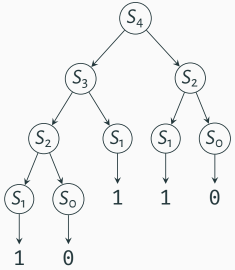

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For algorithmic applications the two important parameters are **size** and **height**.

Example: Random access in time $\mathcal{O}(\text{height})$:

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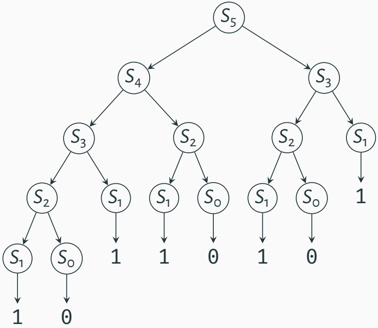

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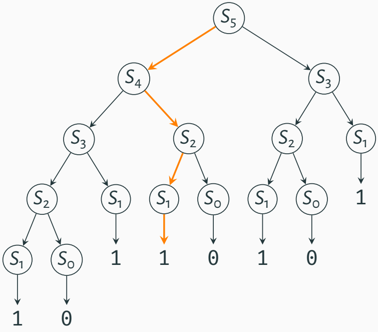

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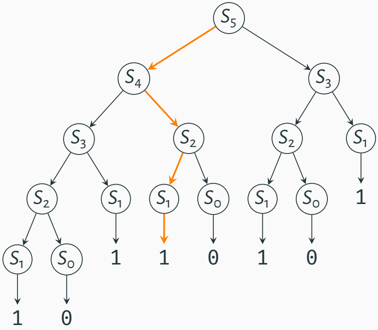

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Logarithmic height

Balancing Theorem (G, Jeż, Lohrey, FOCS 2019, JACM 2021)

Given an SLP $\mathcal{G}$ for a string of length N. One can compute in linear time an equivalent SLP of height $\mathcal{O}(\log N)$ and size $\mathcal{O}(|\mathcal{G}|)$.

$\rightarrow$ previously: $\mathcal{O}(|\mathcal{G}| \cdot \log N)$ [Rytter, 2002; Charikar et al., 2002]

$\rightarrow$ simple solution for random access in $\mathcal{O}(\log N)$ time and **linear** space

Other applications:

- rank and select queries, computing fingerprints, range minimum queries, subsequence matching
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This paper

1. Can we refine the balancing theorem, establishing stronger balancedness properties “for free”? (= $\mathcal{O}(1)$ factor increase)

2. Which algorithmic applications can be obtained using such balancing results?

Compressed pattern matching

Does balancing lead to improved algorithms for compressed pattern matching?

Given an uncompressed pattern P of length m, and a compressed text T of length N and compressed size n.

Question Does P occur in T?

Theorem (Gawrychowksi, 2011)

Compressed pattern matching can be solved in time

- $\mathcal{O}(m + n \cdot \log N)$ for LZ77-compression and for SLPs,
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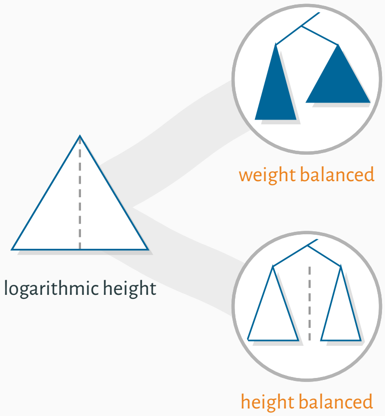
Zoo of balanced SLPs



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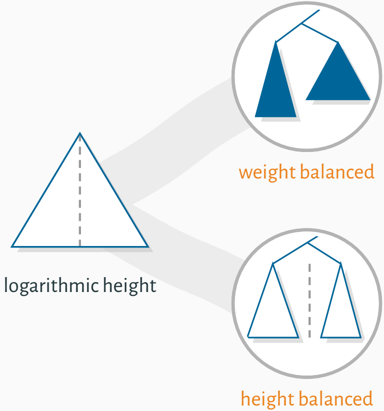

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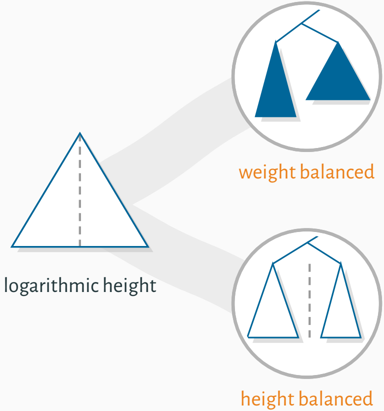

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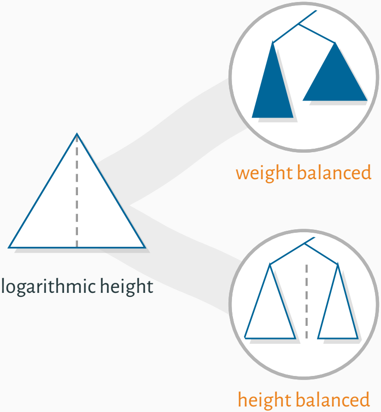

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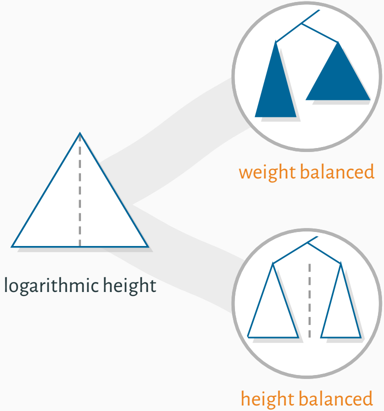

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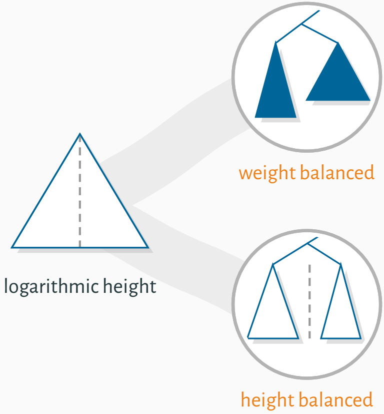

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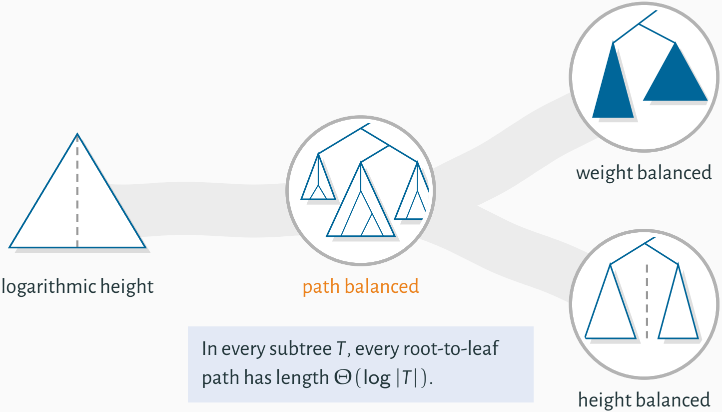
There exist SLPs of size $\mathcal{O}(n)$ such that any equivalent path balanced SLP has size $\Omega(n \log N)$.



In every subtree T, every root-to-leaf path has length $\Theta(\log |T|)$.

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Good News

1. Compressed Pattern Matching

Theorem (G, Gawrychowksi, 2022)

Compressed pattern matching for SLP-compressed texts can be solved in time $\mathcal{O}(m+n)$.

Relies only on logarithmic height SLPs (and new data structures).

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Contracting SLPs

Definition

An SLP is **contracting** if for every rule $A \to \beta_1 \ldots \beta_k$ and every variable β_i we have $|\beta_i| \leqslant |A|/2$.

- Every variable A has height $\mathcal{O}(\log |A|)$ (locally balanced).
- Given a variable A, one can access $A[i]$ in time $\mathcal{O}(\log |A|)$.
- Useful when multiple strings $s_1, \ldots, s_m$ are compressed using a single SLP.

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One can convert an SLP $\mathcal{G}$ in linear time into an equivalent contracting SLP of size $\mathcal{O}(|\mathcal{G}|)$ with rules of constant length.

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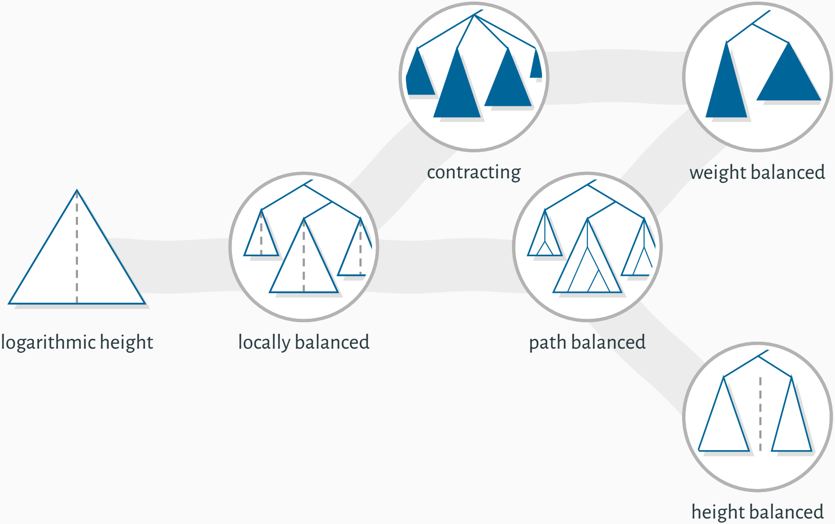
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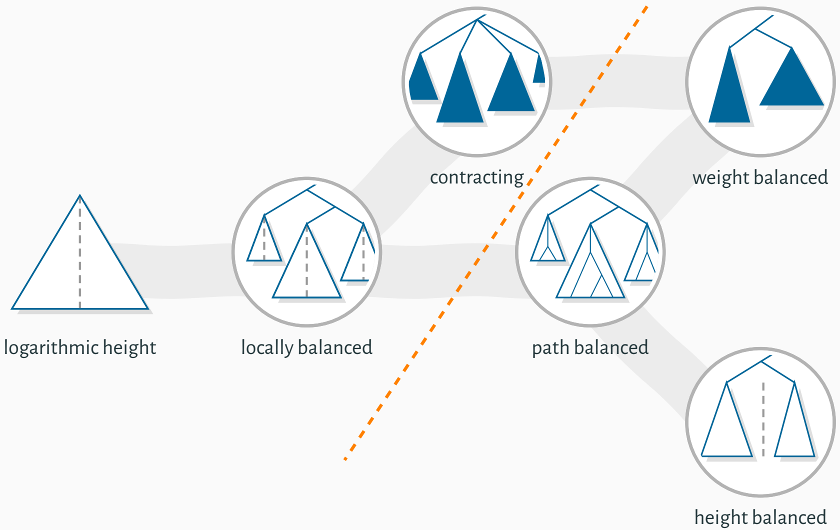
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Proof sketch

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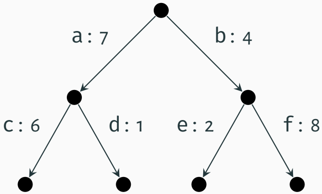
Given a trie T with edges labeled by weighted symbols,
define all prefixes by a contracting SLP.



Possible with a contracting SLP of size $O(|T|)$.

Applications

Finger search

In **finger search** on a (compressed) string we want to support the following operations: [Bille, Christiansen, Cording, Gørtz, 2018]

- `setFinger`(i)
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where d is the distance between i and the finger position,
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Choosing $t = \log^* N$ yields $\mathcal{O}(\log d)$ time for `access`(i) and `moveFinger`(i).

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Navigation on compressed trees

Extend a navigation data structure on FSLP-compressed trees:

Theorem (Reh, Sieber, 2020)

Given a forest SLP for a tree T, one can support in linear space the following navigation steps on T in constant time:

- `parent()` in $\mathcal{O}(1)$ time
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Theorem (Reh, Sieber, 2020; G, 2021)

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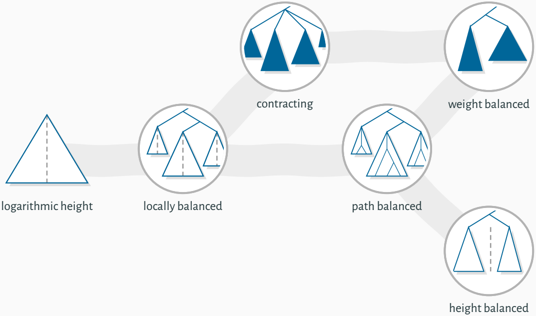
- `parent()` in $\mathcal{O}(1)$ time
- `first_child()`, `last_child()` in $\mathcal{O}(1)$ time
- `next_sibling()`, `prev_sibling()` in $\mathcal{O}(1)$ time
- `get_symbol()` in $\mathcal{O}(1)$ time
- `child(`i`)` in $\mathcal{O}(\log d)$ time

where d is the degree of the current node.

Conclusion

Balancing in grammar-based compression as a preprocessing step that enables fast queries on the compressed data.



Open questions:

Finger search in $\mathcal{O}(\log d)$ time and $\mathcal{O}(|\mathcal{G}|)$ space?
random access for LZ77 in $\mathcal{O}(\log N)$ time and linear space?
Balancing for LZ77/collage systems?

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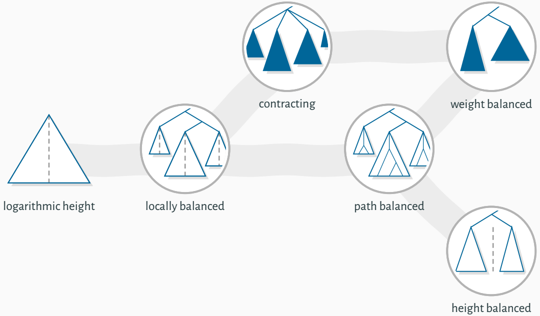

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